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SPACES HOMEOMORPHIC TO $(2^{\alpha})_{\alpha}$. II

BY

H. H. HUNG[1] AND S. NEGREPONTIS[1]

ABSTRACT. Topological characterizations and properties of the spaces $(2^{\alpha})_{\alpha}$, where α is an infinite regular cardinal, are studied; the principal interest lying in the significance that these spaces have in questions of existence of ultrafilters (or of elements of the Stone-Čech compactification of spaces) with special properties. The main results are (a) the characterization theorem of the spaces $(2^{\alpha})_{\alpha}$ in terms of a simple set of conditions, and (b) the α-Baire category property of $(2^{\alpha})_{\alpha}$ and the stability of the class of spaces homeomorphic to $(2^{\alpha})_{\alpha}$ (or to $(\alpha^{\alpha})_{\alpha}$) when taking intersections of at most α open and dense subsets of $(2^{\alpha})_{\alpha}$. Among the applications of these results are the following. Assuming $\alpha^{+} = 2^{\alpha}$, the class of spaces homeomorphic to $(2^{(\alpha^{+})})_{\alpha^{+}}$ includes the space of uniform ultrafilters on α with the $P_{\alpha^{+}}$-topology $(U(\alpha))_{\alpha^{+}}$, its subspaces of good ultrafilters and/or Rudin-Keisler minimal ultrafilters. Assuming $\omega^{+} = 2^{\omega}$ (or in some cases only Martin's axiom), the class of spaces homeomorphic to $(2^{(\omega^{+})})_{\omega^{+}}$ includes the following: The space $(\beta X\setminus X)_{\omega^{+}}$ where X is a noncompact locally compact realcompact space such that $|C(X)| \leq 2^{\omega}$ and its subspaces of $P_{\omega^{+}}$-points of $\beta X\setminus X$ and/or (if X is in addition a metric space without isolated elements) the remote points. In particular the existence of good and/or Rudin-Keisler minimal ultrafilters and the existence of P-points and/or remote points follows always from a Baire category type of argument.

1. **Preliminaries.** The axiom of choice is assumed. Ordinal numbers are denoted by ξ, ζ, η, i. An ordinal coincides with the set of all smaller ordinals, i.e., $\xi < \zeta$ is equivalent to $\xi \in \zeta$. Nevertheless we make the notational distinction between the first ordinal 0 and the empty set $\emptyset$. A cardinal number is an initial ordinal. Cardinals are denoted by $\alpha, \beta, \gamma, \cdots$. $0, 1, \cdots, n, k, \cdots$ denote natural numbers. The first infinite cardinal is ω. The least cardinal greater than β is denoted by β^{+}. The cardinality of the set of all functions from β to α is

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denoted by α^{β}. The generalized continuum hypothesis (G.C.H.) states that $\alpha^{+} = 2^{\alpha}$ if $\omega \leq \alpha$. The cardinality of a set A is denoted by $|A|$. A set of cardinality α and a discrete space of cardinality α is usually identified with α. For a set A we let $\mathcal{P}(A)$, $\mathcal{P}_{\alpha}(A)$ denote the set of all subsets of A, all subsets of A of cardinality less than α, respectively. A cardinal α is regular if it is not equal to the (cardinal) sum of fewer than α smaller than α cardinals; otherwise α is singular. For every infinite cardinal α, α^{+} is a regular cardinal. A cardinal α satisfying the condition $2^{\beta} < \alpha$ if $\beta < \alpha$, is a strong limit cardinal. If α is a regular and a strong limit cardinal then α is called strongly inaccessible. We set $\alpha^{\underline{\beta}} = \Sigma\,\{\alpha^{\gamma}\colon \gamma < \beta\}$. It is known that $\alpha = \alpha^{\underline{\alpha}}$ if and only if $\alpha = 2^{\underline{\alpha}}$ and α is regular. If α is an infinite *regular* cardinal and β any cardinal ≥ 2 then $(\beta^{\underline{\alpha}})^{\underline{\alpha}} = \beta^{\underline{\alpha}}$ (cf. §34 of [1]); thus if α is an infinite regular cardinal we have $\alpha^{\underline{\alpha}} = 2^{\underline{\alpha}}$. If α is strongly inaccessible then $\alpha = \alpha^{\underline{\alpha}}$. For information on cardinal arithmetic the reader is referred to [1], [19].

By a space we mean a completely regular Hausdorff topological space. A zero set in a space X is a set Z such that $Z = \{x \in X\colon f(x) = 0\}$ for some continuous real-valued function f defined on X. The weight $w(X)$ of a space X is the least cardinal α such that X has a base for its topology of cardinality α. The local weight of an element p of X is the least cardinal α such that there is a (filter) base for the (filter of) neighborhoods of p of cardinality α. The Souslin number $S(X)$ of a space X is the least cardinal α such that there is no family of pairwise disjoint nonempty open subsets of X of cardinality α. A space X is α-compact if every open cover of X has a subcover of cardinality less than α; thus ω-compactness coincides with compactness. A space X is a P_{α}-space if the intersection of every family of less than α open subsets of X is an open subset of X; an element p of X is a P_{α}-point if the intersection of a family of less than α neighborhoods of p is a (not necessarily open) neighborhood of p. The usual definitions of P-space, P-point (as given in [8]) coincide with the cases of $P_{\omega^{+}}$-space, $P_{\omega^{+}}$-point, respectively. If α is an infinite singular cardinal then a P_{α}-point is also a $P_{\alpha^{+}}$-point as it is easy to verify. Thus it is no loss of generality to restrict attention to P_{α}-spaces for regular cardinals α.

For $\alpha \geq \omega$ and a space X, a family $\mathcal{B}$ of open subsets of X is an α-subbase (for the topology) of X if the family $\mathcal{C}$ consisting of all intersections of families of elements of $\mathcal{B}$ of cardinality less than α is a base of X. For any space X, and an infinite regular cardinal α, we associate two P_{α}-spaces, denoted $P_{\alpha}(X)$ and X_{α}, as follows: $P_{\alpha}(X)$ is the (possibly empty) subspace of X consisting of all the P_{α}-points of X; and X_{α} is the space with underlying set equal to the underlying set of X and its topology defined by the requirement that the topology of X is an α-subbase for the topology of X_{α}. It is clear that $X_{\omega} = X$, and that if

$\alpha > \omega$ then an α-subbase of X_α is the family of all zero-sets of X. It is clear that the identity mapping $X_\alpha \to X$ is continuous, and that when restricted to $P_\alpha(X)$ it is, in fact, a homeomorphism.

If $\{X_i: i \in I\}$ is a family of topological spaces and $\alpha \geq \omega$, the α-product topology on the product set $\Pi_{i\in I} X_i$ is the topology defined by the condition that the family of all sets of the form $\Pi_{i\in I} G_i$, where G_i is open in X_i and $|\{i \in I: G_i \neq X_i\}| < \alpha$, is a base for this topology. The product set with the α-product topology is denoted $(\Pi_{i\in I} X_i)_{(\alpha)}$; in the case of a power set, we write $(X^I)_{(\alpha)}$. It is easy to see that if α is an infinite regular cardinal and if X_i is a P_α-space for $i \in I$ (in particular, if X_i is discrete for $i \in I$), then $(\Pi_{i\in I} X_i)_\alpha = (\Pi_{i\in I} X_i)_{(\alpha)}$. Thus, for example, $(\beta^\alpha)_\alpha = (\beta^\alpha)_{(\alpha)}$, for any cardinals $\beta \geq 2$ and α infinite regular.

A regular cardinal α is called weakly compact if the space $(2^\alpha)_\alpha$ is α-compact. Various other characterizations of weakly compact cardinals are known (cf. [15], [21], [25], [27]); these are not used in this paper. A weakly compact cardinal is strongly inaccessible.

Let $\Lambda(\alpha)$ denote the set 2^α, ordered lexicographically. Thus, if $p = \langle p_\xi: \xi < \alpha\rangle$, $q = \langle q_\xi: \xi < \alpha\rangle \in \Lambda(\alpha)$, then $p < q$ if $p \neq q$ and, setting $\xi(p, q)$ to be the least ordinal $\xi < \alpha$ such that $p_\xi \neq q_\xi$, we have $p_{\xi(p,q)} = 0 < 1 = q_{\xi(p,q)}$. Also denoted by $\Lambda(\alpha)$ is the space whose underlying set is 2^α and whose (order) topology is determined by the requirement that the open intervals form a base for the topology. The space $\Lambda(\alpha)$ is compact in the order topology. For detailed information on $\Lambda(\alpha)$ the reader is referred to [8], [26].

If $\mathfrak{B}$ is a family of closed nonempty subsets of a space X, we say that X is $\mathfrak{B}$-compact if whenever $\mathcal{C} \subset \mathfrak{B}$ and $\mathcal{C}$ has the finite intersection property (i.e., every finite subset of $\mathcal{C}$ has nonempty intersection), then $\bigcap \mathcal{C} \neq \emptyset$. Thus X is compact if and only if X is $\mathfrak{F}^*$-compact, where $\mathfrak{F}^*$ is the family of all nonempty closed subsets of X.

An open partition $\mathcal{C}$ of a space X is a family of open pairwise disjoint nonempty substes of X whose union is X. If $\mathfrak{B}$, $\mathcal{C}$ are open partitions of X, we say that $\mathfrak{B}$ is a refinement of $\mathcal{C}$ (denoted $\mathcal{C} \prec \mathfrak{B}$) if every element of $\mathfrak{B}$ is a subset of a (necessarily unique) element of $\mathcal{C}$. If X is a P_α-space, if $\{\mathfrak{B}_i: i \in I\}$ is a family of open partitions of X and if $|I| < \alpha$, we let $\bigwedge_{i\in I} \mathfrak{B}_i$ be the family of all nonempty sets of the form $\bigcap_{i\in I} B_i$, where $B_i \in \mathfrak{B}_i$ for $i \in I$. It is clear that $\bigwedge_{i\in I} \mathfrak{B}_i$ is an open partition of X, and that (despite the misleading notation), we have $\mathfrak{B}_j \prec \bigwedge_{i\in I} \mathfrak{B}_i$ for all $j \in I$. We note that if $I = \emptyset$ then $\bigwedge_{i\in I} \mathfrak{B}_i = \{X\}$.

A ramification system is a partially ordered set $\langle A, \preceq\rangle$ with a least element (if it is nonempty) and such that the set

$$P(a) = \{x \in A: x \preceq a \text{ and } x \neq a\}$$

is well ordered by $\preceq$ for $a \in A$. The order of an element $a \in A$ is the order type of $P(a)$, i.e., the unique ordinal isomorphic to the well-ordered set $P(a)$. The order of the ramification system $\langle A, \preceq \rangle$ is the least ordinal ξ, such that the order of $P(a) < \xi$ for all $a \in A$. In this paper we deal with ramification systems $\mathfrak{A}$ whose elements are all subsets of a given set S, and whose partial order $\preceq$ is the reverse set inclusion, i.e., if $A, B \in \mathfrak{A}$ then $A \preceq B$ if and only if $A \supset B$.

For an infinite regular cardinal α the space $(2^\alpha)_\alpha$ has a *canonical* base $\mathfrak{E}$ for its topology such that

$(2^\alpha)_\alpha$ is $\mathfrak{E}$-compact,

$\mathfrak{E}$ is a ramification system under reverse set inclusion,

the set $\mathfrak{E}_\xi$ of elements of $\mathfrak{E}$ of order ξ is an open partition of $(2^\alpha)_\alpha$ for $\xi < \alpha$,

$\mathfrak{E}_\xi = \bigwedge_{\zeta<\xi} \mathfrak{E}_\zeta$ for limit ordinals $\xi < \alpha$, and

if $E \in \mathfrak{E}_\xi$ then E has *exactly two* immediate successors (in $\mathfrak{E}_{\xi+1}$) for $\xi < \alpha$.

In fact, let $E(s) = \{t \in 2^\alpha : t|\xi = s\}$ for $s \in 2^\xi$ and set

$\mathfrak{E}_\xi = \{E(s): s \in 2^\xi\}$ for $\xi < \alpha$, and

$\mathfrak{E} = \bigcup_{\xi<\alpha} \mathfrak{E}_\xi$.

It is clear that $|E_\xi| = 2^{|\xi|}$ for $\xi < \alpha$.

The space $(2^\alpha)_\alpha$ has been studied in [5], [10], [25], [26], [27], [29], [30].

2. We prove in this section the main theorem on the characterization of the space $(2^\alpha)_\alpha$ (Theorem 2.3).

2.1. **Lemma.** *Let X and Y be spaces, let $\mathfrak{A}$ and $\mathfrak{B}$ be subbases of X and Y, respectively, and assume that there is a one-to-one function ϕ from $\mathfrak{A}$ onto $\mathfrak{B}$, such that*

$$\bigcap \mathcal{C} = \emptyset \quad \text{if and only if} \quad \bigcap \phi[\mathcal{C}] = \emptyset$$

for $\mathcal{C} \subset \mathfrak{A}$. Then X is homeomorphic to Y.

Proof. We note that $\bigcap\{\phi(A): A \in \mathfrak{A}$ and $p \in A\}$ consists of a single element of Y for $p \in X$. Indeed, by our assumption on ϕ, the intersection is not empty. If it contains two distinct elements q_0, q_1, there is $B \in \mathfrak{B}$ such that $q_0 \in B$ and $q_1 \notin B$, and hence $B \notin \{\phi(A): A \in \mathfrak{A}$ and $p \in A\}$. It follows that $\phi^{-1}(B) \notin \{A: A \in \mathfrak{A}$ and $p \in A\}$, hence that $\phi^{-1}(B) \cap \bigcap\{A: A \in \mathfrak{A}$ and $p \in A\} = \emptyset$, i.e.,

$$B \cap \bigcap\{\phi(A): A \in \mathfrak{A} \text{ and } p \in A\} = \emptyset,$$

a contradiction, since q_0 is an element of the intersection. We define $f: X \to Y$ by letting $f(p)$ be the unique element of $\bigcap\{\phi(A): A \in \mathfrak{A}$ and $p \in \mathfrak{A}\}$ for $x \in X$. We verify the following statements.

(i) f is one-to-one and onto.

There is a function $g: Y \to X$ defined symmetrically to f and for which we have

$$\begin{aligned}\{f \circ g(q)\} &= \bigcap\{\phi(A): A \in \mathfrak{A} \text{ and } g(q) \in A\}\\ &\subset \bigcap\{\phi(A): A = \phi^{-1}(B),\ B \in \mathfrak{B}, \text{ and } q \in B\}\\ &= \bigcap\{B: B \in \mathfrak{B} \text{ and } q \in B\} = \{q\}, \quad \text{i.e., } f \circ g(q) = q,\end{aligned}$$

for $q \in Y$. It follows that f is onto, and symmetrically that f is one-to-one.

(ii) f is a homeomorphism.

It suffices to prove that $\phi(A) = f[A]$ for $A \in \mathfrak{A}$, since ϕ and f are one-to-one. It is clear that $\phi(A) \supset f[A]$ from the definition of f. By symmetry, $A \supset f^{-1}[\phi(A)]$, i.e., $f[A] \supset \phi(A)$.

The proof of the lemma is complete.

The following notation will be used in §§2 and 3. Let $\mathcal{C}$ be a family of open subsets of X such that $\mathcal{C}$ is a ramification system of order α under reverse set inclusion and let $\mathcal{C}_\xi$ denote the set of elements of $\mathcal{C}$ of order $\xi < \alpha$. If $C \in \mathcal{C}_\xi$ and $\eta < \alpha$ we set

$$\mathcal{S}(C, \eta) = \{D \in \mathcal{C}_{\xi+\eta}: D \subset C\}.$$

In particular, $\mathcal{S}(C, 1)$ denotes the set of immediate successors of C (in the system $\mathcal{C}$). We note that if in addition $\mathcal{C}_\xi$ is an open partition of X for $\xi < \alpha$ then $\mathcal{S}(C, \eta)$ is an open partition of C for $C \in \mathcal{C}$ and $\eta < \alpha$.

2.2. **Lemma.** *Let α be an infinite regular cardinal and let X be a P_α-space without isolated elements and such that X has an α-subbase $\mathfrak{B}$, such that*

X is $\mathfrak{B}$-compact,

$\mathfrak{B} = \bigcup_{\xi<\alpha} \mathfrak{B}_\xi$, and

$\mathfrak{B}_\xi$ is an open partition of X for $\xi < \alpha$.

Then there is a base $\mathcal{C}$ (for the topology) of X, such that

X is $\mathcal{C}$-compact,

$\mathcal{C}$ is a ramification system of order α with respect to reverse set inclusion,

the set $\mathcal{C}_\xi$ of elements of $\mathcal{C}$ of order ξ is an open partition of X, and

$\mathcal{C}_\xi = \bigwedge_{\zeta<\xi} \mathcal{C}_\zeta$ for limit ordinals $\xi < \alpha$.

Hence the space $(2^\alpha)_\alpha$ is the continuous image of X.

Proof. We set

$\mathfrak{A}_\xi = \bigwedge_{\zeta<\alpha} \mathfrak{B}_\zeta$ for $\xi < \alpha$, and

$\mathcal{C} = \bigcup_{\xi<\alpha} \mathfrak{A}_\xi$.

We claim that if $A = \bigcap_{\zeta<\xi} B_\zeta \in \mathfrak{A}_\xi$, where $\xi < \alpha$ and $B_\zeta \in \mathfrak{B}_\zeta$ for $\zeta < \xi$, then

$$(*) \qquad \{B \in \mathcal{C}: B \supset A\} = \left\{\bigcap_{\zeta<\eta} B_\zeta: \eta \le \xi\right\}.$$

Indeed, if $\eta \le \xi$ then $\bigcap_{\zeta<\eta} B_\zeta \supset \bigcap_{\zeta<\xi} B_\zeta = A$. For the converse inclusion, let $B \in \mathcal{C}$ and $B \supset A$. Since $B \in \mathcal{C}$, we have $B = \bigcap_{\zeta<\eta} B'_\zeta \neq \emptyset$, where $\eta < \alpha$ and $B'_\zeta \in \mathcal{B}_\zeta$ for $\zeta < \eta$. If $B'_\zeta \neq B_\zeta$ for some $\zeta < \eta$ and $\zeta < \xi$, then since $B_\zeta, B'_\zeta \in \mathcal{B}_\zeta$ and $\mathcal{B}_\zeta$ is an open partition of X, it follows that $B_\zeta \cap B'_\zeta = \emptyset$, hence that $A = B \cap A = \emptyset$, a contradiction. If now $\eta > \xi$, then $A = \bigcap_{\zeta<\xi} B_\zeta = \bigcap_{\zeta<\xi} B'_\zeta \supset \bigcap_{\zeta<\eta} B'_\zeta = B$, hence $A = B$, and thus we actually have $B = \bigcap_{\zeta<\xi} B_\zeta$ as well, verifying the converse inclusion in this case; if $\eta \le \xi$, then the converse inclusion clearly holds.

We now verify the following statements.

(i) *$\mathcal{C}$ is a base for X.* Let V be a neighborhood of $p \in X$. There is $\beta < \alpha$ and ordinals $\xi(i) < \alpha$ for $i < \beta$ and $B_{\xi(i)} \in \mathcal{B}_{\xi(i)}$, such that

$$p \in \bigcap_{i<\beta} B_{\xi(i)} \subset V.$$

Let $\xi = \sup_{i<\beta} \xi(i)$; then $\xi < \alpha$, since α is regular, and thus it is clear that there is $A \in \mathcal{C}$ such that $p \in A \subset \bigcap_{i<\beta} B_{\xi(i)}$.

(ii) *X is $\mathcal{C}$-compact.* Indeed let $\{A^{(i)}: i \in I\}$ be a family in $\mathcal{C}$ with the finite intersection property. By property (*), this family is a chain under reverse set inclusion. Hence there are $\xi \le \alpha$ and $B_\zeta \in \mathcal{B}_\zeta$ for $\zeta < \xi$ and $\zeta(i) < \xi$ such that $A^{(i)} = \bigcap_{\zeta<\zeta(i)} B_\zeta$ for $i \in I$. Then $\{B_\zeta: \zeta < \xi\}$ has the finite intersection property and, since X is $\mathcal{B}$-compact, we have

$$\emptyset \neq \bigcap_{\zeta<\xi} B_\zeta \subset \bigcap_{\zeta < \sup_{i\in I} \zeta(i)} B_\zeta = \bigcap_{i\in I} \bigcap_{\zeta<\zeta(i)} B_\zeta = \bigcap_{i\in I} A^{(i)}.$$

(iii) *$\mathcal{C}$ is a ramification system of order α under reverse set inclusion.* That $\mathcal{C}$ is a ramification system follows directly from property (*). It is clear that the order of $\mathcal{C}$ is at most α, and since X is a P_α-space without isolated elements and, by (i) above, $\mathcal{C}$ is a base of X, the order of $\mathcal{C}$ cannot be less than α.

(iv) *The set $\mathcal{C}_\xi$ of elements of $\mathcal{C}$ of order ξ is an open partition of X.* Let $p \in X$. The set $\{A \in \mathcal{C}: p \in A\}$ is a well-ordered set under reverse set inclusion since $\mathcal{C}$ is a ramification system, and its order type is α since X is a P_α-space without isolated elements. Hence there is $A \in \mathcal{C}$ such that $p \in A$ and $A \in \mathcal{C}_\xi$, proving that $\mathcal{C}_\xi$ is an open cover of X. Further, if A and B are elements of $\mathcal{C}$ and $A \neq B$, but $A \cap B \neq \emptyset$, then they are comparable and hence they cannot have equal orders.

(v) *$\mathcal{C}_\xi = \bigwedge_{\zeta<\xi} \mathcal{C}_\zeta$ for limit ordinals $\xi < \alpha$.* If $C \in \mathcal{C}_\xi$ then there is an order-isomorphism ϕ from ξ to the set of predecessors of C. Because $C \in \mathcal{C}$, there are $\eta < \alpha$ and $B_i \in \mathcal{B}_i$ for $i < \eta$ such that $C = \bigcap_{i<\eta} B_i$. By property (*) there is $\eta(\zeta) < \eta$ such that $\phi(\zeta) = \bigcap_{i<\eta(\zeta)} B_i$ for $\zeta < \xi$. It is clear that

$\eta(\zeta) < \eta(\zeta')$ if $\zeta < \zeta' < \xi$,

$\phi(\zeta) \in \mathcal{C}_\zeta$ for $\zeta < \xi$, and

$C = \bigcap_{\zeta<\xi} \phi(\zeta) \in \bigwedge_{\zeta<\xi} \mathcal{C}_\zeta$.

Conversely let $C \in \bigwedge_{\zeta<\xi} \mathcal{C}_\zeta$; thus $C = \bigcap_{\zeta<\xi} C_\zeta$ where $C_\zeta \in \mathcal{C}_\zeta$ for $\zeta < \xi$. Since the order of C_ζ is ζ, and $C_{\zeta'} \subsetneq C_\zeta$ if $\zeta < \zeta' < \xi$, the order of C is ξ, i.e., $C \in \mathcal{C}_\xi$.

The proof of the existence of $\mathcal{C}$ with the required properties is complete. It remains to prove that there is a continuous mapping from X onto $(2^\alpha)_\alpha$. Let $\mathfrak{S}$ be the canonical base of $(2^\alpha)_\alpha$. We define a mapping $\phi: \mathcal{C} \to \mathfrak{S}$ recursively, as follows. We let $\phi_0: \mathcal{C}_0 \to \mathfrak{S}_0$ be given by $\phi_0(X) = (2^\alpha)_\alpha$. If ξ be a nonzero limit ordinal $< \alpha$, and if $\phi_\zeta: \mathcal{C}_\zeta \to \mathfrak{S}_\zeta$ has been defined for $\zeta < \xi$, we define $\phi_\xi: \mathcal{C}_\xi \to \mathfrak{S}_\xi$ by

$$\phi_\xi \left(\bigcap_{\zeta<\xi} C_\zeta \right) = \bigcap_{\zeta<\xi} \phi_\zeta(C_\zeta),$$

where $C_\zeta \in \mathcal{C}_\zeta$ for $\zeta < \xi$ and $\bigcap_{\zeta<\xi} C_\zeta \neq \emptyset$.

Let now $\xi < \alpha$ and we define $\phi_{\xi+1}$. Let $C \in \mathcal{C}_\xi$; then $|\mathcal{S}(C, 1)| \geq 2$, and thus there are sets $\mathcal{S}_0(C, 1)$, $\mathcal{S}_1(C, 1)$ such that

$\mathcal{S}(C, 1) = \mathcal{S}_0(C, 1) \cup \mathcal{S}_1(C, 1)$,

$\mathcal{S}_0(C, 1) \neq \emptyset$, $\mathcal{S}_1(C, 1) \neq \emptyset$, and

$\mathcal{S}_0(C, 1) \cap \mathcal{S}_1(C, 1) = \emptyset$.

Let $E_{C,0}$ and $E_{C,1}$ be the two immediate successors of $\phi(C)$ in $\mathfrak{S}_{\xi+1}$. We define $\phi_{\xi+1}$ by the conditions

$\phi_{\xi+1}[\mathcal{S}_0(C, 1)] = \{E_{C,0}\}$, and

$\phi_{\xi+1}[\mathcal{S}_1(C, 1)] = \{E_{C,1}\}$.

Finally, we let $\phi = \bigcup_{\xi<\alpha} \phi_\xi: \mathcal{C} \to \mathfrak{S}$. It is clear that ϕ satisfies the following conditions:

$\phi[\mathcal{C}_\xi] = \mathfrak{S}_\xi$ for $\xi < \alpha$, and

if $C, D \in \mathcal{C}$ and $C \subset D$ then $\phi(C) \subset \phi(D)$.

We define $f: X \to (2^\alpha)_\alpha$ by the condition

$$\{f(p)\} = \bigcap \{\phi(C): C \in \mathcal{C} \text{ and } p \in C\} \quad \text{for } p \in C$$

and we verify the following statements.

(i) *f is well defined.* Indeed if $p \in C$ then $\{C \in \mathcal{C}: p \in C\}$ is a (well-ordered) chain of $\mathcal{C}$ and hence $\{\phi(C): C \in \mathcal{C}$ and $p \in C\}$ is a chain of $\mathfrak{S}$; further, $\{C \in \mathcal{C}: p \in C\} \cap \mathcal{C}_\xi \neq \emptyset$ and hence $\{\phi(C): C \in \mathcal{C}$ and $p \in C\} \cap \mathfrak{S}_\xi \neq \emptyset$ for $\xi < \alpha$. Hence $\bigcap \{\phi(C): C \in \mathcal{C}$ and $p \in C\}$ consists of a single element of $(2^\alpha)_\alpha$, i.e., f is well defined.

(ii) *f is an onto function.* Let $q \in (2^\alpha)_\alpha$. We define a family $\{C_\xi: \xi < \alpha\}$ such that

$C_\xi \in \mathcal{C}_\xi$ for $\xi < \alpha$,

$\{C_\xi: \xi < \alpha\}$ is a chain in $\mathcal{C}$, and

$q \in \phi(C_\xi)$ for $\xi < \alpha$.

We proceed recursively. Let $C_0 = X$. If ξ is a nonzero limit ordinal $< \alpha$ we set $C_\xi = \bigcap_{\zeta<\xi} C_\zeta$. Let now $\xi < \alpha$ and we define $C_{\xi+1}$. Since $q \in \phi(C_\xi) \in \mathcal{E}_\xi$ and $q \in E_{\xi+1}$ for a unique $E_{\xi+1} \in \mathcal{E}_{\xi+1}$ we have that $E_{\xi+1} \subset \phi(C_\xi)$. The properties of ϕ imply that there is $C_{\xi+1} \in \mathcal{C}_{\xi+1}$ such that $\phi(C_{\xi+1}) = E_{\xi+1}$ and $C_{\xi+1} \subset C_\xi$. This completes the recursive definition. We define $p \in X$ by the rule $\{p\} = \bigcap_{\xi<\alpha} C_\xi$. It is clear that $f(p) = q$.

(iii) *f is continuous.* Let $p \in X$, $E \in \mathcal{E}$, and $f(p) \in E$. We must prove that there is $C \in \mathcal{C}$ such that $p \in C$ and $f[C] \subset E$. Let $\xi < \alpha$ be such that $E \in \mathcal{E}_\xi$ and there is $C \in \mathcal{C}_\xi$ such that $\phi(C) = E$. We claim that $f[C] \subset \phi(C)$. Indeed let $p \in C$. Then $\{f(p)\} = \bigcap\{\phi(D): D \in \mathcal{C}$ and $p \in D\} \subset \phi(C)$.

The proof of the lemma is complete.

2.3. **Theorem.** *Let α be an infinite regular cardinal. Let X be a P_α-space without isolated elements and with an α-subbase $\mathfrak{A}$ such that*

$|\mathfrak{A}| \leq 2^{\underline{\alpha}}$,

X is $\mathfrak{A}$-compact,

$\mathfrak{A} = \bigcup_{\xi<\alpha} \mathfrak{A}_\xi$, *and*

$\mathfrak{A}_\xi$ *is an open partition of X for $\xi < \alpha$.*

(a) *If α is not a weakly compact cardinal, then X is homeomorphic to $(2^\alpha)_\alpha$.*

(b) *If α is a weakly compact cardinal, then X is homeomorphic to $(2^\alpha)_\alpha$ if and only if $|\mathfrak{A}_\xi| < \alpha$ for $\xi < \alpha$.*

Proof. From Lemma 2.2 there is a base $\mathcal{C}$ for the topology of X such that

(i) X is $\mathcal{C}$-compact,

(ii) $\mathcal{C}$ is a ramification system of order α with respect to reverse set inclusion,

(iii) the set $\mathcal{C}_\xi$ of elements of $\mathcal{C}$ of order ξ is an open partition of X of cardinality at most $2^{\underline{\alpha}}$ for $\xi < \alpha$, and

(iv) $\mathcal{C}_\xi = \bigwedge_{\zeta<\xi} \mathcal{C}_\zeta$ for limit ordinals $\xi < \alpha$.

We note that from (ii) and (iii) (or from (iv)) we have $\mathcal{C}_0 = \{X\}$; and from (ii) every element of $\mathcal{C}$ has *at least two* immediate successors. The (additional) fact (not given by Lemma 2.2) that $|\mathcal{C}_\xi| \leq 2^{\underline{\alpha}}$ for $\xi < \alpha$ follows from the method of proof of Lemma 2.2, using the equality $(2^{\underline{\alpha}})^{\underline{\alpha}} = 2^{\underline{\alpha}}$ which (as noted in §1) holds for infinite regular cardinals α. We note, as a consequence, that if $C \in \mathcal{C}$ and $\eta < \alpha$ then $2^{|\eta|} \leq |\mathcal{S}(C, \eta)| \leq 2^{\underline{\alpha}}$. We set

$$\alpha_{C,\beta} = |\mathcal{S}(C, \beta)| \quad \text{for } \beta < \alpha.$$

Proof of (a). We define a base $\mathfrak{B}$ for the topology of X such that

$\mathfrak{B} \subset \mathcal{C}$,

$\mathfrak{B}$ satisfies conditions (i), (ii), (iv), and

(v) if $B \in \mathfrak{B}$ then B has exactly $2^{\mathfrak{a}}$ immediate successors (in the system $\mathfrak{B}$).

We proceed recursively. Let $\mathfrak{B}_0 = \{X\}$. If ξ is a nonzero limit ordinal $< \alpha$ we set $\mathfrak{B}_\xi = \bigwedge_{\zeta<\xi} \mathfrak{B}_\zeta$. Let now $\xi < \alpha$ and we define $\mathfrak{B}_{\xi+1}$. It is clear from the recursive assumptions that $|\mathfrak{B}_\xi| = 2^{\mathfrak{a}}$ if $\xi > 0$. There are two (partially overlapping) cases to consider.

Case 1. α is not strongly inaccessible. Thus there is a cardinal $\beta < \alpha$ such that $\alpha \le 2^\beta$. Then $\alpha \le 2^\beta \le \alpha_{B,\beta} \le 2^{\mathfrak{a}}$ for $B \in \mathfrak{B}_\xi$. Let $\phi_{B,\beta}$ be a function of $\alpha_{B,\beta}$ onto α and let $\{C_{B,\beta,i}: i < \alpha_{B,\beta}\}$ be a well-ordering of $\mathcal{S}(B, \beta)$ for $B \in \mathfrak{B}_\xi$. We define

$$\mathfrak{B}_{\xi+1} = \bigcup\{\mathcal{S}(C_{B,\beta,i}, \phi_{B,\beta}(i)): B \in \mathfrak{B}_\xi \text{ and } i < \alpha_{B,\beta}\}.$$

It is easy to verify that $\mathfrak{B}_{\xi+1}$ satisfies the recursive conditions.

Case 2. $\alpha = \alpha^{\mathfrak{a}}$ and α is not a weakly compact cardinal. Thus the space $(2^\alpha)_\alpha$ is not α-compact. Since, according to Lemma 2.2, $(2^\alpha)_\alpha$ is the continuous image of X, it follows that X is not α-compact. In fact every nonempty open-and-closed subset of X is easily seen to satisfy the conditions of the theorem, and hence the conditions of Lemma 2.2, and in particular if $C \in \mathcal{C}$ then C is not α-compact. Since $\mathcal{C}$ is a base for the topology of X, for every $C \in \mathcal{C}$ there is an open cover $\mathfrak{U}_C$ of C such that

$\mathfrak{U}_C \subset \mathcal{C}$, and

no subcover of $\mathfrak{U}_C$ has cardinality less than α.

Since $\alpha = \alpha^{\mathfrak{a}}$, we have $|\mathfrak{U}_C| = \alpha$. We define

$$\mathcal{O}_C = \{B \in \mathfrak{U}_C: \text{there is no } B' \in \mathfrak{U}_C \text{ such that } B \subsetneq B'\},$$

and we prove that $\mathcal{O}_C$ is an open partition of C. Indeed let $p \in C$; since $\mathcal{C}$ is a ramification system under reverse inclusion the family $\{B \in \mathfrak{U}_C: p \in B\}$ is a well-ordered chain in $\mathcal{C}$ and thus there is a least element B_0 of this family. It is clear that $B_0 \in \mathcal{O}_C$ and thus $\mathcal{O}_C$ is a cover of C. Further if $B, B' \in \mathcal{O}_C$ and $B \neq B'$ then the inclusions $B \subset B'$ and $B' \subset B$ both fail (from the definition of $\mathcal{O}_C$) and thus $B \cap B' = \emptyset$.

We note, since $\mathcal{O}_C$ is a subcover of $\mathfrak{U}_C$, that $|\mathcal{O}_C| = \alpha$. We define

$$\mathfrak{B}_{\xi+1} = \bigcup\{\mathcal{O}_B: B \in \mathfrak{B}_\xi\}.$$

It is clear that $\mathfrak{B}_{\xi+1}$ satisfies the recursive conditions.

The recursive definition of the family $\{\mathfrak{B}_\xi: \xi < \alpha\}$ is now complete in both cases. We set $\mathfrak{B} = \bigcup_{\xi<\alpha} \mathfrak{B}_\xi$.

It follows easily from Lemma 2.1 that any two spaces X and Y with bases $\mathfrak{B}$ and $\mathfrak{D}$, respectively, satisfying (the analogues of) conditions (i), (ii), (iii) and

(iv) are homeomorphic. The space $(2^\alpha)_\alpha$ clearly satisfies the conditions of the present theorem. Thus X is homeomorphic to $(2^\alpha)_\alpha$. The proof of part (a) is complete.

Proof of (b). Assume that $|\mathfrak{A}_\xi| = \alpha$ for some $\xi < \alpha$. Then X has an open partition of X of cardinality α and clearly it is not α-compact. Thus X is not homeomorphic to the α-compact space $(2^\alpha)_\alpha$.

For the converse implication assume that $|\mathfrak{A}_\xi| < \alpha$ for $\xi < \alpha$.

If $\alpha = \omega$ then X is a totally disconnected space without isolated elements and with a countable base for its topology. In addition we note that X is compact; this is a consequence of the fact that X is $\mathcal{C}$-compact, by making use of the classical D. König's theorem. Thus X is homeomorphic to the Cantor set 2^ω $(= (2^\omega)_\omega)$.

Let now $\alpha > \omega$ and let X and X' be two spaces satisfying the conditions of the theorem. Since α is strongly inaccessible, there are bases $\mathcal{C}$ and $\mathcal{C}'$ for the topologies of X and X', respectively, satisfying conditions (i) to (iv) above and such that $|\mathcal{C}_\xi| < \alpha$ and $|\mathcal{C}'_\xi| < \alpha$ for $\xi < \alpha$.

We define bases $\mathfrak{B}$ and $\mathfrak{B}'$ for the topologies of X and X', respectively, such that

$\mathfrak{B} \subset \mathcal{C}$ and $\mathfrak{B}' \subset \mathcal{C}'$,

$\mathfrak{B}$ and $\mathfrak{B}'$ satisfy (the analogues of) conditions (i), (ii), (iv) above, and

$|\mathfrak{B}_\xi| < \alpha$ and $|\mathfrak{B}'_\xi| < \alpha$ for $\xi < \alpha$,

and a function $\phi: \mathfrak{B} \to \mathfrak{B}'$ such that

ϕ is one-to-one and onto,

$\phi[\mathfrak{B}_\xi] = \mathfrak{B}'_\xi$ for $\xi < \alpha$, and

if $B_0, B_1 \in \mathfrak{B}$ and $B_0 \subset B_1$ then $\phi(B_0) \subset \phi(B_1)$.

We proceed recursively. Let $\mathfrak{B}_0 = \{X\}$, $\mathfrak{B}'_0 = \{X'\}$, and $\phi_0: \mathfrak{B}_0 \to \mathfrak{B}'_0$ the unique mapping. If ξ is a nonzero limit ordinal $< \alpha$ we set

$\mathfrak{B}_\xi = \bigwedge_{\zeta<\xi} \mathfrak{B}_\zeta$,

$\mathfrak{B}'_\xi = \bigwedge_{\zeta<\xi} \mathfrak{B}'_\zeta$, and

$\phi_\xi(\bigcap_{\zeta<\xi} B_\zeta) = \bigcap_{\zeta<\xi} \phi_\xi(B_\zeta)$ if $B_\zeta \in \mathfrak{B}_\zeta$ for $\zeta < \xi$ and $\bigcap_{\zeta<\xi} B_\zeta \neq \emptyset$.

Let now $\xi < \alpha$ and we define $\mathfrak{B}_{\xi+1}$, $\mathfrak{B}'_{\xi+1}$, and $\phi_{\xi+1}$. The recursive assumptions clearly imply that $|\mathfrak{B}_\xi| = |\mathfrak{B}'_\xi| < \alpha$. For $B \in \mathfrak{B}_\xi$ we set $B' = \phi_\xi(B)$ and we define recursively ordinals $\eta_{B,n} < \alpha$ and $\eta_{B',n} < \alpha$ such that

$\eta_{B,0} = 1$,

$\eta_{B,n} \leq \eta_{B'n} \leq \eta_{B,n+1}$, and

$2 \leq |\mathcal{S}(B, \eta_{B,n})| \leq |\mathcal{S}(B', \eta_{B',n})| \leq |\mathcal{S}(B, \eta_{B,n+1})|$ for $n < \omega$.

We set

$$\eta_B = \sup_{n<\omega} \eta_{B,n} = \sup_{n<\omega} \eta_{B',n},$$

and it is clear that $|\mathcal{S}(B, \eta_B)| = |\mathcal{S}(B', \eta_B)| < \alpha$ for $B \in \mathfrak{B}_\xi$. We let $\phi_{B,\xi+1}$ be any one-to-one function of $\mathcal{S}(B, \eta_B)$ onto $\mathcal{S}(B', \eta)_B$ and we define

$\mathcal{B}_{\xi+1} = \bigcup\{\mathcal{S}(B, \eta_B): B \in \mathcal{B}_{\xi}\}$,

$\mathcal{B}'_{\xi+1} = \bigcup\{\mathcal{S}(B', \eta_B): B \in \mathcal{B}_{\xi}\}$, and

$\phi_{\xi+1} = \bigcup\{\phi_{B,\xi+1}: B \in \mathcal{B}_{\xi}\}$.

The recursive definitions are complete. We set

$\mathcal{B} = \bigcup_{\xi<\alpha} \mathcal{B}_{\xi}$,

$\mathcal{B}' = \bigcup_{\xi<\alpha} \mathcal{B}'_{\xi}$, and

$\phi = \bigcup_{\xi<\alpha} \phi_{\xi}$.

It follows from Lemma 2.1 that X and X' are homeomorphic and in particular that X is homeomorphic to $(2^{\alpha})_{\alpha}$.

The proof of the theorem is complete.

3. In this section we examine the α-Baire category property of $(2^{\alpha})_{\alpha}$ and the stability of the class of spaces homeomorphic to $(2^{\alpha})_{\alpha}$ (or to $(\alpha^{\alpha})_{\alpha}$) when taking intersections of families of at most α open and dense subsets of $(2^{\alpha})_{\alpha}$ (Theorem 3.5). As a consequence of the methods developed in §2 (together with Lemma 3.1) we have, for example, that $((2^{\underline{\alpha}})^{\alpha})_{\alpha}$ is homeomorphic to $(\alpha^{\alpha})_{\alpha}$ for all infinite regular cardinals α (Theorem 3.2(a)) and that α is weakly compact if and only if $(2^{\alpha})_{\alpha}$ is not homeomorphic to $(\alpha^{\alpha})_{\alpha}$ (Corollary 3.4).

3.1. Lemma. *Let α be an infinite regular cardinal and let X be a $(P_{\alpha}$-$)$ space (without isolated elements) having a base $\mathcal{B}$ for its topology such that*

(i) *X is $\mathcal{B}$-compact,*

(ii) *$\mathcal{B}$ is a ramification system of order α under reverse set inclusion,*

(iii) *the set $\mathcal{B}_{\xi}$ of elements of order ξ is an open partition of X of cardinality at most $2^{\underline{\alpha}}$ for $\xi < \alpha$,*

(iv) *$\mathcal{B}_{\xi} = \bigwedge_{\zeta<\xi} \mathcal{B}_{\zeta}$ for limit ordinals $\xi < \alpha$, and*

(v) *if $B \in \mathcal{B}$ then B has at least α immediate successors (with respect to the partial order of the ramification system $\mathcal{B}$).*

Then X is homeomorphic to $(\alpha^{\alpha})_{\alpha}$.

Proof. We define a base $\mathcal{C}$ of X such that $\mathcal{C} \subset \mathcal{B}$,

$\mathcal{C}$ satisfies (i), (ii), (iii), (iv), and

(vi) if $C \in \mathcal{C}$ then C has *exactly* $2^{\underline{\alpha}}$ immediate successors (with respect to the order of the ramification system $\mathcal{C}$).

We proceed recursively. Let $\mathcal{C}_0 = \mathcal{B}_0 = \{X\}$. If ξ is a nonzero limit ordinal $< \alpha$, we set

$$\mathcal{C}_{\xi} = \bigwedge_{\zeta<\xi} \mathcal{C}_{\zeta}.$$

Let now $\xi < \alpha$ and we define $\mathcal{C}_{\xi+1}$. Clearly from the recursive assumptions, we have $|\mathcal{C}_\xi| = 2^{\mathfrak{a}}$ for $\xi > 0$. For every $C \in \mathcal{C}_\xi$, we have $\alpha \leq |\mathcal{S}(C, 1)| \leq 2^{\mathfrak{a}}$. Let $\{B_{C,i}: i < |\mathcal{S}(C, 1)|\}$ be a well-ordering of $\mathcal{S}(C, 1)$ and let $\phi_C: |\mathcal{S}(C, 1)| \to \alpha$ be an onto mapping. We define

$$\mathcal{C}_{\xi+1} = \bigcup \{\mathcal{S}(B_{C,i}, \phi_C(i)): C \in \mathcal{C}_\xi,\ i < |\mathcal{S}(C, 1)|\}.$$

The recursive definition is complete.

It follows easily from Lemma 2.1 that any two spaces X and Y with bases $\mathcal{C}$ and $\mathfrak{D}$, respectively, satisfying (the analogues of) conditions (i), (ii), (iii), (iv), and (vi) are homeomorphic.

The space $(\alpha^\alpha)_\alpha$ has a *canonical* base $\mathcal{E}'$ for its topology satisfying conditions (i) to (v) of the present lemma, and in fact

the set $\mathcal{E}'_\xi$ of elements of order ξ has cardinality $\alpha^{|\xi|}$ for $\xi < \alpha$, and
if $E \in \mathcal{E}'$ then E has *exactly* α immediate successors.
In fact let $E'(s) = \{t \in \alpha^\alpha: t|\xi = s\}$ for $s \in \alpha^\xi$ and set
$\mathcal{E}'_\xi = \{E'(s): s \in \alpha^\xi\}$ for $\xi < \alpha$, and
$\mathcal{E}' = \bigcup_{\xi<\alpha} \mathcal{E}'_\xi$.

This completes the proof of the lemma.

3.2. Theorem. *Let α be an infinite regular cardinal.*

(a) *If $\alpha \leq \beta_\xi \leq 2^{\mathfrak{a}}$ for $\xi < \alpha$ then the space $(\Pi_{\xi<\alpha} \beta_\xi)_\alpha$ is homeomorphic to $(\alpha^\alpha)_\alpha$.*

In particular, if $\alpha \leq \beta \leq 2^{\mathfrak{a}}$ then $(\beta^\alpha)_\alpha$ is homeomorphic to $(\alpha^\alpha)_\alpha$; thus $((2^{\mathfrak{a}})^\alpha)_\alpha$ is homeomorphic to $(\alpha^\alpha)_\alpha$.

(b) *If $2 \leq \beta_\xi < \alpha$ for $\xi < \alpha$ then the space $(\Pi_{\xi<\alpha} \beta_\xi)_\alpha$ is homeomorphic to $(2^\alpha)_\alpha$.*

In particular, if $2 \leq \beta < \alpha$, then $(\beta^\alpha)_\alpha$ is homeomorphic to $(2^\alpha)_\alpha$.

(c) *If α is not weakly compact and if $2 \leq \beta_\xi \leq 2^{\mathfrak{a}}$ for $\xi < \alpha$ then the space $(\Pi_{\xi<\alpha} \beta_\xi)_\alpha$ is homeomorphic to $(2^\alpha)_\alpha$ (and also to $(\alpha^\alpha)_\alpha$).*

(d) *If $1 \leq \beta_\xi < \alpha$ for $\xi < \alpha$ then the space $(\Pi_{\xi<\alpha} 2^{\beta_\xi})_\alpha$ is homeomorphic to $(2^\alpha)_\alpha$.*

In particular, if $1 \leq \beta < \alpha$ then $((2^\beta)^\alpha)_\alpha$ is homeomorphic to $(2^\alpha)_\alpha$.

Proof. (a) follows from Lemma 3.1.

(b) follows from Theorem 2.3 (a) and (b).

(c) follows from Theorem 2.3 (a).

(d) If $2^{\beta_\xi} < \alpha$ for $\xi < \alpha$, then the statement follows from part (b).

If $2^{\beta_\xi} \geq \alpha$ for some $\xi < \alpha$ then α is not strongly inaccessible, in particular, α is not weakly compact and $2^{\beta_\xi} \leq 2^{\mathfrak{a}}$ for $\xi < \alpha$. This case follows then from part (c).

3.3. Corollary. *Let* α *be an infinite regular cardinal. The following statements are equivalent.*

(a) $\alpha = 2^{\underline{\alpha}}$.

(b) *If* $\beta > \alpha$ *then* $(2^{\alpha})_{\alpha}$ *is not homeomorphic to* $(\beta^{\alpha})_{\alpha}$.

(c) *If* $\beta > \alpha$ *then* $(\alpha^{\alpha})_{\alpha}$ *is not homeomorphic to* $(\beta^{\alpha})_{\alpha}$.

Proof. (a) $\rightarrow$ (b). Assume $\alpha = 2^{\underline{\alpha}}$ and $\beta > \alpha$. Then it is clear that $(2^{\alpha})_{\alpha}$ is α^{+}-compact and hence β-compact, while $(\beta^{\alpha})_{\alpha}$ is not β-compact. Alternatively we note that the Souslin number of $(2^{\alpha})_{\alpha}$ is at most α^{+} (Theorem 2.3 of [5]), while the Souslin number of $(\beta^{\alpha})_{\alpha}$ is obviously at least β^{+}.

(b) $\rightarrow$ (a). If $\alpha < 2^{\underline{\alpha}}$ then there is $\beta < \alpha$ such that $2^{\beta} > \alpha$. By Theorem 3.2 (d) we have $((2^{\beta})^{\alpha})_{\alpha}$ is homeomorphic to $(2^{\alpha})_{\alpha}$.

(a) $\rightarrow$ (c). Assume $\alpha = 2^{\underline{\alpha}}$ and $\beta > \alpha$. Then we have $\alpha = \alpha^{\underline{\alpha}}$ hence that $(\alpha^{\alpha})_{\alpha}$ is α^{+}-compact and thus β-compact, while $(\beta^{\alpha})_{\alpha}$ is not β-compact. Alternatively we note that $S((\alpha^{\alpha})_{\alpha}) \leq \alpha^{+}$ (Theorem 2.3 of [5]), while $S((\beta^{\alpha})_{\alpha}) \geq \beta^{+}$.

(c) $\rightarrow$ (a) follows from Theorem 3.2 (a).

3.4. Corollary. *Let* α *be an infinite regular cardinal. Then* α *is weakly compact if and only if* $(2^{\alpha})_{\alpha}$ *is not homeomorphic to* $(\alpha^{\alpha})_{\alpha}$.

Proof. If $(\alpha^{\alpha})_{\alpha}$ is homeomorphic to $(2^{\alpha})_{\alpha}$ then $(2^{\alpha})_{\alpha}$ is not α-compact and hence α is not weakly compact. Conversely if α is not weakly compact the statement follows from Theorem 2.3 (a).

A space X is said to have the *α-Baire category property* if the intersection of a family of at most α open and dense subsets of X is dense.

Part (a) of Theorem 3.5 below is due to Sikorski (Theorem (xv), §4 of [30]) and it is included here for completeness. Part (c) of Theorem 3.5 is proved in Kuratowski [18] for $\alpha = \omega$ and attributed there to Mazurkiewicz.

The following notation will be used below. If $\mathfrak{B}$ is a family of open nonempty subsets of X and G is a dense subset of X we set

$$\mathfrak{B} \upharpoonright G = \{B \cap G : B \in \mathfrak{B}\}.$$

3.5. Theorem. *Let* α *be an infinite regular cardinal.*

(a) *The space* $(2^{\alpha})_{\alpha}$ *has the* α*-Baire category property.*

(b) *If* α *is not weakly compact, the intersection of any family of at most* α *open and dense subsets of* $(2^{\alpha})_{\alpha}$ *is homeomorphic to* $(2^{\alpha})_{\alpha}$.

(c) *The intersection* G *of any family of at most* α *open subsets of* $(2^{\alpha})_{\alpha}$ *such that* G *and* $(2^{\alpha})_{\alpha} \backslash G$ *are both dense in* $(2^{\alpha})_{\alpha}$ *is homeomorphic to* $(\alpha^{\alpha})_{\alpha}$.

Proof. (a) Let $G = \bigcap_{\xi < \alpha} G_{\xi}$, where G_{ξ} is open and dense in $(2^{\alpha})_{\alpha}$ for $\xi < \alpha$. It suffices to prove that $E \cap G \neq \emptyset$ for all $E \in \mathfrak{S}$, where $\mathfrak{S}$ denotes the canonical base of $(2^{\alpha})_{\alpha}$. We define a family $\{E_{\xi}: \xi < \alpha\}$ such that

$E_0 = E$,

$E_{\xi} \in \mathfrak{E}$ for $\xi < \alpha$,

$E_{\xi+1} \subset E_{\xi} \cap G_{\xi}$ for $\xi < \alpha$, and

$E_{\xi} = \bigcap_{\zeta<\xi} E_{\zeta}$ for nonzero limit ordinals $\xi < \alpha$.

We proceed recursively. We define $E_0 = E$; if ξ is a nonzero limit ordinal $< \alpha$ we set $E_{\xi} = \bigcap_{\zeta<\xi} E_{\zeta}$. The properties of the canonical base $\mathfrak{E}$ imply that $E_{\xi} \in \mathfrak{E}$. Let now $\xi < \alpha$ and we define $E_{\xi+1}$. Since $E_{\xi} \in \mathfrak{E}$ and E_{ξ} is open and nonempty and G_{ξ} is open and dense in $(2^{\alpha})_{\alpha}$, it follows that $E_{\xi} \cap G_{\xi}$ is open and nonempty. The fact that $\mathfrak{E}$ is a base implies that there is $E_{\xi+1} \in \mathfrak{E}$ such that $E_{\xi+1} \subset E_{\xi} \cap G_{\xi}$. This completes the recursive definition. Since $(2^{\alpha})_{\alpha}$ is $\mathfrak{E}$-compact, we have

$$\varnothing \neq \bigcap_{\xi<\alpha} E_{\xi} = \bigcap_{\xi<\alpha} E_{\xi+1} \subset E \cap \bigcap_{\xi<\alpha} G_{\xi} = E \cap G.$$

(b) Let $G = \bigcap_{\xi<\alpha} G_{\xi}$, where $\{G_{\xi}: \xi < \alpha\}$ is a family of open and dense subsets of $(2^{\alpha})_{\alpha}$. With no loss of generality, because of part (a), we assume in addition that $G_{\xi} \subset G_{\zeta}$ for $\zeta < \xi < \alpha$.

We define families $\mathfrak{B}_{\xi}$ for $\xi < \alpha$ such that

$\mathfrak{B}_{\xi} \subset \bigcup_{\xi\le\zeta<\alpha} \mathfrak{E}_{\zeta}$, and

$\mathfrak{B}_{\xi}$ is an open partition of G_{ξ}.

Let $\mathfrak{U}_{\xi}$ be an open cover of G_{ξ} such that $\mathfrak{U}_{\xi} \subset \bigcup_{\xi\le\zeta<\alpha} \mathfrak{E}_{\zeta}$. Let $\mathfrak{B}_{\xi} = \{E \in \mathfrak{U}_{\xi}$: there is no $E' \in \mathfrak{U}_{\xi}$ such that $E \subsetneq E'\}$. We prove that $\mathfrak{B}_{\xi}$ is an open partition of G_{ξ} for $\xi < \alpha$. Indeed let $p \in G_{\xi}$; since $\mathfrak{E}$ is a ramification system under reverse inclusion, the family $\{E \in \mathfrak{U}_{\xi}: p \in E\}$ is a well-ordered chain in $\mathfrak{E}$ and thus there is a least element E_0 of this family. It is clear that $E_0 \in \mathfrak{B}_{\xi}$ and thus $\mathfrak{B}_{\xi}$ is a cover of G_{ξ}. Further if $E, E' \in \mathfrak{B}_{\xi}$ and $E \neq E'$ then the inclusions $E \subset E'$ and $E' \subset E$ both fail (from the definition of $\mathfrak{B}_{\xi}$) and thus $E \cap E' = \varnothing$.

We set $\mathfrak{B} = \bigcup_{\xi<\alpha} (\mathfrak{B}_{\xi} \restriction G)$ and we prove that $\mathfrak{B}$ satisfies the conditions of Theorem 2.3 (a). It suffices to verify the following statements.

(i) *$\mathfrak{B}$ is a base for the topology of G.* Let $p \in G$ and let U be a neighborhood of p in $(2^{\alpha})_{\alpha}$. There are $\xi < \alpha$ and $E \in \mathfrak{E}_{\xi}$ such that $p \in E \subset U$. Let B_{ξ} be the unique element of $\mathfrak{B}_{\xi}$ such that $p \in B_{\xi}$. Since $\mathfrak{B}_{\xi} \subset \bigcup_{\xi\le\zeta<\alpha} \mathfrak{E}_{\zeta}$ it follows that $B_{\xi} \subset E$.

(ii) *G is $\mathfrak{B}$-compact.* Indeed let $\mathfrak{C}$ be a subfamily of $\mathfrak{B}$ with the finite intersection property. Let $\beta = |\mathfrak{C}|$ and let $\{C_i: i < \beta\}$ be a well-ordering of $\mathfrak{C}$ such that $C_i \neq C_j$ if $i < j < \beta$. Thus $C_i = B_i \cap G$ where $B_i \in \bigcup_{\xi\le\alpha} \mathfrak{B}_{\xi} \subset \mathfrak{E}$ for $i < \beta$; then $\{B_i: i < \beta\}$ has the finite intersection property and hence $\bigcap_{i<\beta} B_i \neq \varnothing$. There are two possibilities to consider.

Case 1. $\beta < \alpha$. Then $\bigcap_{i<\beta} B_i$ is open and nonempty and $\bigcap_{i<\beta} C_i = \bigcap_{i<\beta} B_i \cap G \neq \emptyset$.

Case 2. $\beta = \alpha$. Then $B_i \in \mathfrak{B}_{\xi(i)}$ for some $\xi(i) < \alpha$ and $|\{\xi(i): i < \beta\}| = \alpha$ and hence the set $\{\xi(i): i < \beta\}$ is cofinal in α. It follows that $\bigcap_{i<\beta} B_i \subset \bigcap_{i<\beta} G_{\xi(i)} = \bigcap_{\xi<\alpha} G_\xi = G$ and thus again $\bigcap_{i<\beta} C_i = \bigcap_{i<\beta} B_i \neq \emptyset$.

The proof of part (b) is complete.

(c) Let $G = \bigcap_{\xi<\alpha} G_\xi$, where $\{G_\xi: \xi < \alpha\}$ is a family of open and dense subsets of $(2^\alpha)_\alpha$ and $(2^\alpha)_\alpha \backslash G$ is dense in $(2^\alpha)_\alpha$. With no loss of generality, because of part (a), we assume in addition that $G_\xi \subset G_\zeta$ for $\zeta < \xi < \alpha$.

We define sets H_ξ and families $\mathfrak{B}_\xi$ for $\xi < \alpha$ such that

(i) $H_0 = G_0$ and $\mathfrak{B}_0 = \{G_0\}$;

(ii) H_ξ is open and dense in $(2^\alpha)_\alpha$ for $\xi < \alpha$;

(iii) $H_\xi \subset H_\zeta$ for $\zeta < \xi < \alpha$;

(iv) $H_\xi = \bigcap_{\zeta<\xi} H_\zeta$ for nonzero limit ordinals $\xi < \alpha$;

(v) $G \subset H_{\xi+1} \subset G_\xi$ for $\xi < \alpha$;

(vi) $\mathfrak{B}_\xi \subset \bigcup_{\xi\leq\zeta<\alpha} \mathfrak{S}_\zeta$ for $0 < \xi < \alpha$;

(vii) $\mathfrak{B}_\xi$ is an open partition of H_ξ for $\xi < \alpha$;

(viii) $\mathfrak{B}_\zeta \restriction H_\xi \prec \mathfrak{B}_\xi$ for $\zeta < \xi < \alpha$;

(ix) $\mathfrak{B}_\xi = \bigwedge_{\zeta<\xi} (\mathfrak{B}_\zeta \restriction H_\xi)$ for nonzero limit ordinals $\xi < \alpha$; and

(x) if $B \in \mathfrak{B}_\zeta$ and $\zeta < \xi < \alpha$ then $H_\xi \cap B$ is a *proper* subset of B.

We proceed recursively. We let H_0 and $\mathfrak{B}_0$ be defined by (i).

If ξ is a nonzero limit ordinal $< \alpha$ we define H_ξ and $\mathfrak{B}_\xi$ by (iv) and (ix), respectively. We verify the (relevant) recursive conditions for ξ. For condition (ii) we note that H_ξ is open because X is a P_α-space, and H_ξ is dense by part (a); (iii) is obvious; for (vi) let $B \in \mathfrak{B}_\xi$. Then $B = \bigcap_{\zeta<\xi} (B_\zeta \cap H_\xi)$ where $B_\zeta \in \mathfrak{B}_\zeta$ and thus $B_\zeta \in \bigcup_{\zeta\leq\eta<\alpha} \mathfrak{S}_\eta$ for $0 < \zeta < \xi$. We note that

$$B = \bigcap_{\zeta<\xi} (B_\zeta \cap H_\xi) = \bigcap_{\zeta<\xi} B_\zeta \cap H_\xi$$

$$= \bigcap_{\zeta<\xi} B_\zeta \cap \bigcap_{\zeta<\xi} H_\zeta = \bigcap_{\zeta<\xi} (B_\zeta \cap H_\zeta) = \bigcap_{\zeta<\xi} B_\zeta,$$

and hence, since the nonempty decreasing intersection of a family of less than α elements of the canonical base $\mathfrak{S}$ is in $\mathfrak{S}$, we have that $B \in \mathfrak{S}$; it is clear that in fact $B \in \bigcup_{\xi\leq\zeta<\alpha} \mathfrak{S}_\zeta$. Condition (vii) follows from the fact that X is a P_α-space; (viii) is clear; and for (x) we note that if $B \in \mathfrak{B}_\zeta$ and $\zeta < \xi < \alpha$ then the recursive assumptions imply that $B \cap H_\xi \subset B \cap H_{\zeta+1} \subsetneq B$.

Let $\xi < \alpha$ and we define $H_{\xi+1}$ and $\mathfrak{B}_{\xi+1}$. We note that $G_\xi \cap H_\xi$ is open and dense in $(2^\alpha)_\alpha$ and contains G. Because $(2^\alpha)_\alpha \backslash G$ is dense in $(2^\alpha)_\alpha$ it follows that for $B \in \mathfrak{B}_\xi \restriction (G_\xi \cap H_\xi)$ there is ξ_B such that

$\xi < \xi_B < \alpha$, and

$B \cap G_{\xi_B}$ is a *proper* subset of B.

We define

$$H_{\xi+1} = \bigcup\{B \cap G_{\xi_B} : B \in \mathfrak{B}_\xi \restriction (G_\xi \cap H_\xi)\}.$$

Let $\mathfrak{U}_{\xi+1}$ be an open cover of $H_{\xi+1}$ such that

$\mathfrak{U}_{\xi+1} \subset \bigcup_{\xi<\zeta<\alpha} \mathfrak{E}_\zeta$ and

$\mathfrak{B}_\xi \restriction H_{\xi+1} \prec \mathfrak{U}_{\xi+1}$.

We define

$$\mathfrak{B}_{\xi+1} = \{E \in \mathfrak{U}_{\xi+1}: \text{there is no } E' \in \mathfrak{U}_{\xi+1} \text{ such that } E \subsetneq E'\}.$$

The only condition that requires verification is (vii) and this is proved as the corresponding statement in part (b); the details are omitted. This completes the recursive definitions.

We define $\mathfrak{B} = \bigcup_{\xi<\alpha} (\mathfrak{B}_\xi \restriction G)$. As in part (b), we prove that

$\mathfrak{B}$ is a base of G, and

G is $\mathfrak{B}$-compact.

Further we prove that if $\xi < \alpha$ and $C \in \mathfrak{B}_\xi \restriction G$ then C has *at least* α immediate successors in $\mathfrak{B}_{\xi+1} \restriction G$. Indeed $C = B \cap G$ for a unique $B \in \mathfrak{B}_\xi$; by (x) the set $H_{\xi+1} \cap B$ is a proper subset of B. Because $H_{\xi+1} \cap B$ is open and dense in B, and $\mathfrak{B}_{\xi+1}$ is an open partition of $H_{\xi+1}$ refining $\mathfrak{B}_\xi \restriction H_{\xi+1}$ it follows that

$$|\{D \in \mathfrak{B}_{\xi+1}: D \subset H_{\xi+1} \cap B\}| \geq \alpha.$$

Indeed otherwise since X is a P_α-space, the set $\bigcup\{D \in \mathfrak{B}_{\xi+1}: D \subset H_{\xi+1} \cap B\}$ would be a closed subset of B, contradicting the fact that it is a proper and dense subset of B.

Then $D \cap C \neq \emptyset$ for $D \in \mathfrak{B}_{\xi+1}$ with $D \subset H_{\xi+1} \cap B$ because D is open and nonempty and C is dense in $H_{\xi+1} \cap B$ (since G is dense in $(2^\alpha)_\alpha$). Thus C has at least α immediate successors in $\mathfrak{B}_{\xi+1} \restriction G$.

The conditions of Lemma 3.1 are satisfied proving that G is homeomorphic to $(\alpha^\alpha)_\alpha$.

The proof of the theorem is complete.

We remark that the assumption on the nonweak compactness of the cardinal α in Theorem 3.5 (b) is essential; it is clear that the statement fails for $\alpha = \omega$ (and in fact for every weakly compact cardinal α).

4. **Applications.** The results of §§2–3 (especially Theorems 2.3 and 3.5) will be used now in a large number of cases. The main application of Theorem 2.3 is Theorem 4.1 below which, for any infinite regular cardinal α, gives

sufficient (and necessary if $\alpha = \alpha^{\langle\alpha\rangle}$) conditions on a compact space X in terms of the weight and local weights of X in order that X_{α} is homeomorphic to $(2^{\alpha})_{\alpha}$. The principal corollaries to Theorem 4.1 are contained in the statements of Theorem 4.8 (if α is any cardinal such that $\alpha^{+} = 2^{\alpha}$ then $(U(\alpha))_{\alpha^+}$ is homeomorphic to $(2^{(\alpha^+)})_{\alpha^+}$) and Theorem 4.15 (= Corollary 3.3 of [4]) (if $\omega^{+} = 2^{\omega}$ and X is a non-compact locally compact realcompact space such that $|C(X)| \leq 2^{\omega}$ then $(\beta X\backslash X)_{\omega^+}$ is homeomorphic to $(2^{(\omega^+)})_{\omega^+}$). These identifications open the way to applications of the Baire category type of theorem (3.5) to the spaces $(U(\alpha))_{\alpha^+}$ and $(\beta X\backslash X)_{\omega^+}$ involving good, Rudin-Keisler minimal ultrafilters on α and P-points, remote points of $\beta X\backslash X$.

The applications of Theorem 2.3 and its corollary 4.8 are limited to the systematic use of the Baire category property of these spaces. However, the spaces $(2^{\alpha})_{\alpha}$ and $(\alpha^{\alpha})_{\alpha}$ form the higher cardinal analogues of the Cantor set and the space of irrationals respectively and thus they serve as the fundamental spaces for the analogues to higher cardinalities of a large part of the classical descriptive set theory (sometimes depending on the size of the cardinal α). We do not know if, but consider it quite possible that, these higher cardinality analogues of the classical descriptive set theory when used in the corresponding spaces of ultrafilters can produce similarly useful results.

The following result has been proved for $\alpha = \omega^{+} = 2^{\omega}$ in Theorem 3.2 of [4].

4.1. **Theorem.** *Let α be an infinite regular cardinal and let X be a compact (and totally disconnected if $\alpha = \omega$) space.*

If X is such that

(a) *the weight of X is (at most) α, and*

(b) *the local weight of every element of X is (at least) α*

then X_{α} is homeomorphic to $(2^{\alpha})_{\alpha}$. Conversely, if $\alpha = \alpha^{\langle\alpha\rangle}$ and X_{α} is homeomorphic to $(2^{\alpha})_{\alpha}$ then X satisfies conditions (a) *and* (b).

Proof. If $\alpha = \omega$ we let $\{B_n : n < \omega\}$ be a base for X consisting of open-and-closed sets and we set $\mathfrak{A}_n = \{B_n, X\backslash B_n\}$ for $n < \omega$ and $\mathfrak{A} = \bigcup_{n<\omega} \mathfrak{A}_n$. We note that $|\mathfrak{A}_n| \leq 2 < \omega$ for $n < \omega$.

If $\alpha > \omega$ we let $\{B_{\xi} : \xi < \alpha\}$ be a family of zero-sets of X such that $\{\text{Int}\, B_{\xi} : \xi < \alpha\}$ is a base of X (where Int B_{ξ} denotes the interior of B_{ξ} in X). We have used the standard fact that every base of X contains a subfamily which is a base of X itself and of cardinality $w(X)$. It is clear that $\{B_{\xi} : \xi < \alpha\}$ is an α-subbase for X_{α}. For $\xi < \alpha$ let f_{ξ} be a continuous real-valued function defined on X such that $B_{\xi} = \{x \in X : f_{\xi}(x) = 0\}$. We set

$$\mathfrak{A}_{\xi} = \{f_{\xi}^{-1}(\{t\}) : t \text{ in the image of } f_{\xi}\}$$

for $\xi < \alpha$, and $\mathfrak{A} = \bigcup_{\xi<\alpha} \mathfrak{A}_\xi$. Then $\mathfrak{A}$ consists of open-and-closed subsets of X_α (since a zero-set is closed in X and thus in X_α, and is a G_δ-set in X and thus an open set in X_α). Further $\mathfrak{A}_\xi$ is an open partition of X_α and $|\mathfrak{A}_\xi| \leq 2^\omega \leq 2^{\underline{\alpha}}$ for $\xi < \alpha$. Since $\{B_\xi : \xi < \alpha\} \subset \mathfrak{A}$ then $\mathfrak{A}$ is an α-subbase of X_α. By condition (b) X_α is a P_α-space without isolated elements. Since X is a compact and $\mathfrak{A}$ consists of closed nonempty subsets of X, it follows that X_α is $\mathfrak{A}$-compact.

If α is not weakly compact then $|\mathfrak{A}_\xi| \leq 2^{\underline{\alpha}}$ for $\xi < \alpha$ and by Theorem 2.3 (a) the space X_α is homeomorphic to $(2^\alpha)_\alpha$. If α is weakly compact, then either $\alpha = \omega$ in which case $|\mathfrak{A}_\xi| < \omega$ for $\xi < \alpha$, or $\alpha > \omega$ in which case $|\mathfrak{A}_\xi| \leq 2^\omega < \alpha$ for $\xi < \alpha$. It follows from Theorem 2.3 (b) that X_α is homeomorphic to $(2^\alpha)_\alpha$.

For the (partial) converse assume that X is compact (and totally disconnected if $\alpha = \omega$), that $\alpha = \alpha^{\underline{\alpha}}$ and that X_α is homeomorphic to $(2^\alpha)_\alpha$. It is clear that X satisfies condition (b). Condition (a) will follow from the existence of an embedding of X into the power space $[0, 1]^\alpha$, since then we will have $w(X) \leq w([0, 1]^\alpha) = \alpha$. The statement is trivial if $\alpha = \omega$ and thus we assume $\alpha > \omega$.

Let $\phi: (2^\alpha)_\alpha \rightarrow X_\alpha$ be a homeomorphism. If $B \in \phi[\mathfrak{S}]$ where $\mathfrak{S}$ denotes, as usual, the canonical base of $(2^\alpha)_\alpha$ then

$$B = \bigcup_{i \in I_B} B_i \quad \text{and} \quad B_i = \bigcap_{j \in J_{B,i}} Z_{ij} \quad \text{for } i \in I_B,$$

where

Z_{ij} is a zero-set of X for $i \in I_B$ and $j \in J_{B,i}$,

$|J_{B,i}| < \alpha$ for $i \in I$, and

$|I_B| \leq \alpha$.

We define

$$\mathfrak{A} = \{Z_{ij} : i \in I_B,\ j \in J_{B,i}, \text{ and } B \in \phi[\mathfrak{S}]\}.$$

Then we have

$|\mathfrak{A}| = \alpha$,

$\mathfrak{A}$ is a family of zero-sets of X, and

if $p, q \in X$ and $p \neq q$ then there are $Z_0, Z_1 \in \mathfrak{A}$ such that $Z_0 \cap Z_1 = \emptyset$ and $p \in Z_0$, $q \in Z_1$.

Indeed the first two properties are obvious; to prove the third property let $p, q \in X$ and $p \neq q$. There are $B_0, B_1 \in \phi[\mathfrak{S}]$ such that $p \in B_0$, $q \in B_1$ and $B_0 \cap B_1 = \emptyset$. Then there are indices $i_0 \in I_{B_0}$ and $i_1 \in I_{B_1}$ such that $p \in B_{i_0}$ and $q \in B_{i_1}$. The compactness of X implies that there are $n, m < \omega$ and indices $j_{0,0}, \cdots, j_{0,n} \in J_{B_0,i_0}$ and $j_{1,0}, \cdots, j_{1,m} \in J_{B_1,i_1}$ such that

$$(Z_{j_{0,0}} \cap \cdots \cap Z_{j_{0,n}}) \cap (Z_{j_{1,0}} \cap \cdots \cap Z_{j_{1,m}}) = \emptyset,$$

establishing the third of the properties.

For every pair $Z_0, Z_1 \in \mathcal{A}$ such that $Z_0 \cap Z_1 = \emptyset$ there is a continuous function f_{Z_0,Z_1} defined on X with values in the interval $[0, 1]$ such that $f_{Z_0,Z_1} = 0$ on Z_0 and $f_{Z_0,Z_1} = 1$ on Z_1. Thus the family

$$\mathcal{S} = \{f_{Z_0,Z_1} : Z_0, Z_1 \in \mathcal{A} \text{ and } Z_0 \cap Z_1 = \emptyset\}$$

is a family of continuous functions from X to $[0, 1]$ such that $|\mathcal{S}| = \alpha$ and $\mathcal{S}$ separates elements of X. The embedding lemma (cf. Kelley [16]) and the compactness of X imply the existence of an embedding of X into $[0, 1]^{\alpha}$. The proof of the theorem is complete.

We do not know if condition (a) of the above theorem can be replaced by the weaker condition.

(a′) $w(X) \leq 2^{\underline{\alpha}}$

(it being understood that condition (b) is sharpened to the condition that every element of X has local weight exactly α).

4.2. **Corollary.** (a) *If* $\alpha = \alpha^{\underline{\alpha}}$ *then* $\Lambda(\alpha)_{\alpha}$ *is homeomorphic to* $(2^{\alpha})_{\alpha}$.

(b) $P_{\omega}(\Lambda(\omega)) = P_{\omega}(2^{\omega}) = 2^{\omega}$.

(c) *If* $\alpha = \alpha^{\underline{\alpha}} > \omega$ *then* $P_{\alpha}(\Lambda(\alpha))$ *is homeomorphic to* $(\alpha^{\alpha})_{\alpha}$.

Proof. (a) follows directly from Theorem 4.1, while (b) follows from the fact that $\Lambda(\omega) = 2^{\omega}$. To prove (c) we note, by part (a) that $\Lambda(\alpha)_{\alpha}$ is homeomorphic to $(2^{\alpha})_{\alpha}$ and that

$$P_{\alpha}(\Lambda(\alpha)) = \Lambda(\alpha)_{\alpha} \setminus \text{set of non-}P_{\alpha}\text{-points of } \Lambda(\alpha).$$

It is well known that if $p \in \Lambda(\alpha)$ is a non-P_{α}-point of $\Lambda(\alpha)$ then p is eventually constant (as a function on α to 2). Hence the set of non-P_{α}-points of $\Lambda(\alpha)$ is of cardinality $\Sigma_{\xi<\alpha} 2^{|\xi|} = 2^{\underline{\alpha}} = \alpha$. For every non-$P_{\alpha}$-point p, the set $\Lambda(\alpha)_{\alpha}\setminus\{p\}$ is open in $\Lambda(\alpha)_{\alpha}$ and it is dense in $\Lambda(\alpha)_{\alpha}$, since the local weight at any element of $\Lambda(\alpha)$ is α. Furthermore, the set $\Lambda(\alpha)_{\alpha}\setminus P_{\alpha}(\Lambda(\alpha))$ is dense in $\Lambda(\alpha)_{\alpha}$, because the intersection of a family of less than α open subsets of $\Lambda(\alpha)$ contains an open interval of $\Lambda(\alpha)$ and hence non-P_{α}-points. Thus from Theorem 3.5 (c), $P_{\alpha}(\Lambda(\alpha))$ is homeomorphic to $(\alpha^{\alpha})_{\alpha}$.

If $\alpha = \alpha^{\underline{\alpha}}$ then (according to the results and definitions given in [11], [12], [22]) there is a (unique) α-homeogeneous-universal Boolean algebra of cardinality α, whose Stone space we denote by $S(\alpha)$. The space $S(\alpha)$ has the α-Baire category property and $P_{\alpha}(S(\alpha))$ is the intersection of a family of α open and dense subsets of $S(\alpha)$ (Theorem 3.5 of [23]). The following corollary is a more careful statement of the results given without proof in Theorem 3.8 of [23].

4.3. Corollary. (a) *If* $\alpha = \alpha^{\textcircled{a}}$ *then* $(S(\alpha))_\alpha$ *is homeomorphic to* $(2^\alpha)_\alpha$.
(b) $P_\omega(S(\omega)) = P_\omega(2^\omega) = 2^\omega$.
(c) *If* $\alpha = \alpha^{\textcircled{a}} > \omega$ *then* $P_\alpha(S(\alpha))$ *is homeomorphic to* $(\alpha^\alpha)_\alpha$.

Proof. (a) The space $S(\alpha)$ is compact, totally disconnected of weight α and of local weight α at each of its elements (cf. Theorem 1.7 of [23]).

(b) We note that $S(\omega)$ is (homeomorphic to) 2^ω.

(c) By part (a) the space $(S(\alpha))_\alpha$ is homeomorphic to $(2^\alpha)_\alpha$. We note that the identity function $(S(\alpha))_\alpha \to S(\alpha)$ is a homeomorphism when it is restricted to the subspace $P_\alpha(S(\alpha))$. Furthermore $(S(\alpha))_\alpha \backslash P_\alpha(S(\alpha))$ is dense in $(S(\alpha))_\alpha$ and $P_\alpha(S(\alpha))$ is the intersection of (at most) α open subsets of $S(\alpha)$ (and hence of $(S(\alpha))_\alpha$), each of which is dense in $S(\alpha)$, and in fact in $(S(\alpha))_\alpha$. Indeed

$$P_\alpha(S(\alpha)) = \bigcap \{S(\alpha)\backslash \mathrm{Bd}\ U\colon U \text{ a subset of } S(\alpha) \text{ equal to the union of} < \alpha \text{ open-and-closed subsets of } S(\alpha)\}$$

(where Bd U denotes the boundary of U). The result now follows from Theorem 3.5 (c).

Our aim is to prove Theorem 4.8. The auxiliary results 4.4 to 4.7 are known and their statements and some of the proofs are included here for completeness. We also need some notation and definitions, as given below.

We denote by βX the Stone-Čech compactification of a space X. If $f\colon X \to Y$ is a continuous function we denote by $\bar{f}\colon \beta X \to \beta Y$ the unique continuous Stone extension of f to βX. If α is a cardinal number then $\beta(\alpha)$ is the set of ultrafilters on α with the Stone topology (considered as the Stone space of the Boolean algebra $\mathcal{P}(\alpha)$). If $\mathcal{F} \subset \mathcal{P}(\alpha)$ we say that $\mathcal{F}$ has the uniform finite intersection property if $|A_0 \cap \cdots \cap A_n| = \alpha$ for $n < \omega$ and $A_k \in \mathcal{F}$ for $k \le n$. An ultrafilter p on α is uniform if $|A| = \alpha$ for $A \in p$. The set of uniform ultrafilters on α (considered as a subspace of $\beta(\alpha)$) is denoted by $U(\alpha)$. If $A \in \mathcal{P}(\alpha)$ we set $\hat{A} = \mathrm{cl}_{\beta(\alpha)} A \cap U(\alpha)$ (where $\mathrm{cl}_{\beta(\alpha)} A$ denotes the closure of A in $\beta(\alpha)$).

Two ultrafilters p, q on α are isomorphic (denoted $p \approx q$) if there is a permutation π of α (i.e., a one-to-one function of α onto α) such that $\bar{\pi}(p) = q$; the equivalence classes under (ultrafilter) isomorphism are the types of $\beta(\alpha)$ (which is denoted $\mathrm{T}(\beta(\alpha))$) and the quotient equivalence function is denoted $\tau\colon \beta(\alpha) \to \mathrm{T}(\beta(\alpha))$. The Rudin-Keisler partial preorder $\preceq$ on $\beta(\alpha)$ is the binary relation on $\beta(\alpha)$ defined by: $p \preceq q$ if an only if there is $f \in \alpha^\alpha$ such that $\bar{f}(q) = p$. Because $p \approx q$ if $p \preceq q$ and $q \preceq p$, the resulting Rudin-Keisler order is defined on $\mathrm{T}(\beta(\alpha))$ (cf. [3], [28], [17]). A uniform ultrafilter p on α is uniformly selective if for every $f \in \alpha^\alpha$ there is $A \in p$ such that either $f|A$ is one-to-one or $|f[A]| < \alpha$. It is easy to see that $p \in U(\alpha)$ has minimal type in $\tau[U(\alpha)]$ if and only if p is uniformly selective. We denote by $\mathrm{RK}(\alpha)$ the set of

$p \in U(\alpha)$ such that $\tau(p)$ is minimal in $\tau[U(\alpha)]$ (in the Rudin-Keisler order).

The following lemma, involving a diagonal argument, is a slight generalization of Lemma 3.2 in [24].

4.4. Lemma. *Let α be an infinite cardinal, let $\mathcal{F}$ be a family of subsets of α of cardinality at most α, and having the uniform finite intersection property, and let $d = \{d_\zeta: \zeta < \alpha\}$ be a partition of α. Then there is a subset F of α such that*
$\mathcal{F} \cup \{F\}$ has the uniform finite intersection property, and
$|\{\zeta < \alpha: |F \cap d_\zeta| > 1\}| < \alpha$.

Proof. With no loss of generality, we assume that $\mathcal{F}$ is closed under finite intersections. Let $\{E_\xi: \xi < \alpha\}$ be a well-ordering of $\mathcal{F}$. There are two cases to consider.

Case 1. There is $\xi < \alpha$, such that $|\{\zeta < \alpha: |E_\xi \cap d_\zeta| > 1\}| < \alpha$; we then set $F = E_\xi$.

Case 2. $|\{\zeta < \alpha: |E_\xi \cap d_\zeta| > 1\}| = \alpha$ for every $\xi < \alpha$. We let

$$A_\xi = \{\zeta < \alpha: |E_\xi \cap d_\zeta| > 1\},$$

and thus $|A_\xi| = \alpha$ for $\xi < \alpha$. By Lemma 4A of [13], there is a family $\{B_\xi: \xi < \alpha\}$ of subsets of α such that $B_\xi \subset A_\xi$, $|B_\xi| = \alpha$, and $B_\xi \cap B_\eta = \emptyset$ for $\xi < \eta < \alpha$. We choose $p_{\xi,\zeta} \in E_\xi \cap d_\zeta$ for $\zeta \in B_\xi$, $\xi < \alpha$, and we set $F = \{p_{\xi,\zeta}: \zeta \in B_\xi, \xi < \alpha\}$. Then, $E_\xi \cap F = \{p_{\xi,\zeta}: \zeta \in B_\xi\}$, so $|E_\xi \cap F| = \alpha$ for $\xi < \alpha$. Further, if $\zeta < \alpha$ and $\zeta \notin \bigcup_{\xi<\alpha} B_\xi$, then $F \cap d_\zeta = \emptyset$; if $\zeta < \alpha$ and $\zeta \in B_\xi$ for some (unique) $\xi < \alpha$, then $F \cap d_\zeta = \{p_{\xi,\zeta}\}$. Thus, in this case $|F \cap d_\zeta| \le 1$ for all $\zeta < \alpha$.

4.5. Corollary (Theorem 8.3 in Blass [2]). *Let α be an infinite cardinal, let $f \in \alpha^\alpha$ and set*

$$D_f = \bigcup\{\hat{A}: A \in \mathcal{P}(\alpha) \text{ and either } f|A \text{ is one-to-one or } |f[A]| < \alpha\}.$$

Then D_f is an open and dense subset of $(U(\alpha))_{\alpha^+}$.

Proof. Clearly D_f is open in $U(\alpha)$, hence in $(U(\alpha))_{\alpha^+}$. That D_f is a dense subset of $(U(\alpha))_{\alpha^+}$ is a direct consequence of Lemma 4.4.

Let α, β be cardinals, let ϕ, ψ be functions on $\mathcal{P}_\omega(\beta)$ into $\mathcal{P}(\alpha)$, and let p be an ultrafilter on α. We require the following definitions:

ϕ is monotone if $\phi(F) \subset \phi(G)$ for $F, G \in \mathcal{P}_\omega(\beta)$ and $G \subset F$;

ϕ is multiplicative if $\phi(F \cup G) = \phi(F) \cap \phi(G)$ for $F, G \in \mathcal{P}_\omega(\beta)$;

$\psi \le \phi$ if $\psi(F) \subset \phi(F)$ for $F \in \mathcal{P}_\omega(\beta)$.

An ultrafilter p is α^+-good if for every monotone function $\phi: \mathcal{P}_\omega(\alpha) \to p$ there is a multiplicative function $\psi: \mathcal{P}_\omega(\alpha) \to p$, such that $\psi \le \phi$. (Here we consider that $p \subset \mathcal{P}(\alpha)$.)

These definitions and the next lemma are due to Keisler [13].

4.6. **Lemma** (Lemma 4C in Keisler [13]). *Let α be an infinite cardinal, let $\mathcal{F}$ be a family of subsets of α of cardinality at most α, and let ϕ be a monotone function from $\mathcal{P}_\omega(\alpha)$ into $\mathcal{P}(\alpha)$, such that*

$\mathcal{F} \cup \phi[\mathcal{P}_\omega(\alpha)]$ has the uniform finite intersection property.

Then there is a function $\psi: \mathcal{P}_\omega(\alpha) \to \mathcal{P}(\alpha)$ such that

$\psi \leq \phi$,

ψ is multiplicative, and

$\mathcal{F} \cup \psi[\mathcal{P}_\omega(\alpha)]$ has the uniform finite intersection property.

If $\phi: \mathcal{P}_\omega(\alpha) \to \mathcal{P}(\alpha)$ is a monotone function, we set

$$U_\phi = \bigcap \{\widehat{\phi(F)}: F \in \mathcal{P}_\omega(\alpha)\}$$

(where $\hat{A}$ denotes $\mathrm{cl}_{\beta(\alpha)} A \cap U(\alpha)$ for $A \in \mathcal{P}(\alpha)$).

4.7. **Corollary** (Theorem 8.1 in Blass [2]). *Let α be an infinite cardinal, let $\phi: \mathcal{P}_\omega(\alpha) \to \mathcal{P}(\alpha)$ be a monotone function, and set*

$$E_\phi = ((U(\alpha))_{\alpha^+} \setminus U_\phi)$$
$$\cup \bigcup \{U_\psi: \psi \text{ is a multiplicative function}, \psi: \mathcal{P}_\omega(\alpha) \to \mathcal{P}(\alpha), \text{ and } \psi \leq \phi\}.$$

Then E_ϕ is an open and dense subset of $(U(\alpha))_{\alpha^+}$.

Proof. Each U_ψ is dense in $(U(\alpha))_{\alpha^+}$, because it is equal to the intersection of at most α open-and-closed subsets of $U(\alpha)$; hence E_ϕ is open in $(U(\alpha))_{\alpha^+}$.

To prove that E_ϕ is dense in $(U(\alpha))_{\alpha^+}$, let $\{V_\xi: \xi < \alpha\}$ be a family of open-and-closed subsets of $U(\alpha)$, such that $V = \bigcap_{\xi<\alpha} V_\xi \neq \emptyset$, and we are to prove that $V \cap E_\phi \neq \emptyset$. Let $A_\xi \in \mathcal{P}(\alpha)$ be such that $\hat{A}_\xi = V_\xi$ for $\xi < \alpha$. Then the family $\mathcal{F} = \{A_\xi: \xi < \alpha\}$ is a family of subsets of α of cardinality at most α and with the uniform finite intersection property. There are two possibilities to consider.

Case 1. $\mathcal{F} \cup \phi[\mathcal{P}_\omega(\alpha)]$ has the uniform finite intersection property.

By Lemma 4.6, there is a multiplicative function $\psi: \mathcal{P}_\omega(\alpha) \to \mathcal{P}(\alpha)$ such that $\psi \leq \phi$ and $\mathcal{F} \cup \psi[\mathcal{P}_\omega(\alpha)]$ has the uniform finite intersection property. Thus

$$\emptyset \neq V \cap U_\psi \subset V \cap E_\phi.$$

Case 2. $\mathcal{F} \cup \phi[\mathcal{P}_\omega(\alpha)]$ does not have the uniform finite intersection property. Thus $V \cap U_\phi = \emptyset$ and $V \subset (U(\alpha))_{\alpha^+} \setminus U_\phi \subset E_\phi$.

As a consequence of Theorems 3.5 (a) and 4.8 (b) if $\alpha^+ = 2^\alpha$ then $(U(\alpha))_{\alpha^+}$ has the α^+-Baire category property. This has been proved directly by Blass in [2]. The existence of α^+-good ultrafilters on α was first proved by Keisler in [13] assuming $\alpha^+ = 2^\alpha$, and by Kunen [17] in general. The existence of α^+-good ultrafilters on α which in addition are $P(\alpha)$-points was proved in [24], assuming $\alpha^+ = 2^{\alpha}$. A slight modification of this argument yields the existence of α^+-good and uniformly selective ultrafilters on α, assuming $\alpha^+ = 2^{\alpha}$. A Baire category type of proof on the existence of α^+-good and uniformly selective ultrafilters on α, still assuming $\alpha^+ = 2^{\alpha}$, was first given by Blass in [2]. The following result, one of the principal applications of the methods developed in §§2–3, provides a topological description of these spaces of ultrafilters in the P_{α^+}-topology and assuming $\alpha^+ = 2^{\alpha}$.

4.8. Theorem. *Let α be an infinite cardinal.*

(a) *If 2^{α} is a regular cardinal and no uniform ultrafilter on α has a filter base of cardinality less than 2^{α}, then*

$$(U(\alpha))_{2^{\alpha}} \text{ is homeomorphic to } (2^{(2^{\alpha})})_{2^{\alpha}}.$$

(b) *If $\alpha^+ = 2^{\alpha}$, then each of the spaces $(U(\alpha))_{\alpha^+}$, $(\mathrm{RK}(\alpha))_{\alpha^+}$, $(G(\alpha))_{\alpha^+}$, $(\mathrm{RK}(\alpha) \cap G(a))_{\alpha^+}$ is homeomorphic to $(2^{(\alpha^+)})_{\alpha^+}$.*

Proof. (a) is a direct application of Theorem 4.1 for the cardinal 2^{α}.

(b) The statement about $(U(\alpha))_{\alpha^+}$ follows from part (a), since α^+ is regular, and no uniform ultrafilter on α has a filter base of cardinality at most α.

We verify that

(i) $(\mathrm{RK}(\alpha))_{\alpha^+} = \bigcap\{D_f: f \in \alpha^{\alpha}\}$, and

(ii) $(G(\alpha))_{\alpha^+} = \bigcap\{E_{\phi}: \phi$ monotone function, $\phi: \mathcal{P}_{\omega}(\alpha) \to \mathcal{P}(\alpha)\}$ where the sets D_f, E_{ϕ} are defined in Corollaries 4.5, 4.7, respectively.

To prove (i), let p be a uniform ultrafilter on α, whose type is Rudin-Keisler minimal in $\tau[U(\alpha)]$, and let $f \in \alpha^{\alpha}$. Then there is $A \in p$ such that either $f|A$ is one-to-one or $|f[A]| < \alpha$; hence $p \in \hat{A} \subset D_f$. Conversely, let $p \in \bigcap\{D_f: f \in \alpha^{\alpha}\}$ and let $f \in \alpha^{\alpha}$. Since $p \in D_f$, then $p \in \hat{A}$, i.e., $A \in p$, for some $A \in \mathcal{P}(\alpha)$ such that either $f|A$ is one-to-one or $|f[A]| < \alpha$; hence, the type of p is Rudin-Keisler minimal in $\tau[U(\alpha)]$.

To prove (ii), let $p \in (G(\alpha))_{\alpha^+}$ and let ϕ be a monotone function, ϕ: $\mathcal{P}_{\omega}(\alpha) \to \mathcal{P}(\alpha)$. If $\phi(F) \notin p$ for some $F \in \mathcal{P}_{\omega}(\alpha)$, then $\alpha \backslash \phi(F) \in p$, and hence

$$p \in (U(\alpha))_{\alpha^+} \backslash \widehat{\phi(F)} \subset E_{\phi}.$$

If $\phi(F) \in p$ for $F \in \mathcal{P}_{\omega}(\alpha)$, then, because p is a α^+-good, there is a multiplicative function $\psi: \mathcal{P}_{\omega}(\alpha) \to p$, such that $\psi \le \phi$. Then $p \in \widehat{\psi(F)}$ for $F \in \mathcal{P}_{\omega}(\alpha)$, i.e., $p \in U_{\psi} \subset E_{\phi}$. Thus, in any case, $p \in E_{\phi}$ for all monotone functions ϕ. Conversely, let $p \in \bigcap\{E_{\phi}: \phi$ monotone, $\phi: \mathcal{P}_{\omega}(\alpha) \to \mathcal{P}(\alpha)\}$, and let ϕ: $\mathcal{P}_{\omega}(\alpha) \to p$ be a monotone function. Then $p \in E_{\phi}$ and also $p \in U_{\phi}$; hence, there is some multiplicative function $\psi: \mathcal{P}_\omega(\alpha) \to \mathcal{P}(\alpha)$ such that $\psi \leq \phi$, for which $p \in U_\psi$. Then $\psi(F) \in p$ for $F \in \mathcal{P}_\omega(\alpha)$, i.e., $\psi: \mathcal{P}_\omega(\alpha) \to p$. This proves that $p \in (G(\alpha))_{\alpha^+}$.

The remaining statements of part (b) now follow from the already proved fact that $(U(\alpha))_{\alpha^+}$ is homeomorphic to $(2^{(\alpha^+)})_{\alpha^+}$, together with Theorem 3.5 (b) (since certainly α^+ is not weakly compact; alternatively we could prove easily that Theorem 3.5 (c) can be used), Lemmas 4.5 and 4.7, and the statements (i) and (ii) just proved.

The proof of the theorem is complete.

We now turn to a number of applications, similar in spirit but topological in content, concerning subsets of $\beta X \setminus X$ for certain spaces X.

We recall that a space is σ-compact if it is the union of a sequence of compact sets.

Definitions. A space X is *realcompact* if for every $p \in \beta X \setminus X$ there is a zero-set Z of βX such that $p \in Z \subset \beta X \setminus X$.

A space X is *pseudocompact* if every continuous real-valued function defined on X is bounded.

We denote by $C(X)$ the set of continuous real-valued functions defined on a space X. If $Y \subset X$ then Y is *C-embedded* if for $f \in C(Y)$ there is $g \in C(X)$ such that $f \subset g$.

4.9. **Theorem.** *Let X be a space and consider the following conditions on X.*

(a) *X is a noncompact σ-compact space;*

(b) *X is a noncompact realcompact space;*

(c) *X is not pseudocompact;*

(d) *there is a subset N of X such that N is homeomorphic to ω and is C-embedded in X;*

(e) *there is a subset N of X such that N is homeomorphic to ω and a closed C^*-embedded subset of X; and*

(f) $|\beta X \setminus X| \geq 2^{2^\omega}$.

Then (a) $\to$ (b) $\to$ (c) $\to$ (d) $\to$ (e) $\to$ (f).

All these implications are proved in [8].

The following simple lemma provides a family of spaces to which Theorem 4.15 below applies.

4.10. **Lemma.** *If X is a σ-compact space and $w(X) \leq 2^\omega$ then $|C(X)| \leq 2^\omega$.*

Proof. (a) Assume first that X is a compact space of weight at most 2^ω. There is an embedding e of X into $[0, 1]^{(2^\omega)}$. Since X is compact, $e[X]$ is C-embedded in $[0, 1]^{(2^\omega)}$; it is known (cf. [5], [16]) that the space $[0, 1]^{(2^\omega)}$ is separable (i.e., its density character is ω) and hence clearly $|C([0, 1]^{(2^\omega)})| = 2^\omega$. Thus

$$|C(X)| \le |C[0, 1]^{(2^\omega)}| = 2^\omega.$$

We remark that if $X \ne \emptyset$ then in fact $|C(X)| = 2^\omega$.

(b) Let $\{K_n : n < \omega\}$ be a sequence of compact subsets of X whose union is X. We define a function

$$\phi: C(X) \to \prod_{n<\omega} C(K_n)$$

by $\phi(f) = \langle f|K_n : n < \omega \rangle$. Then ϕ is a well-defined one-to-one function, and thus we have (making use of part (a))

$$|C(X)| \le \prod_{n<\omega} |C(K_n)| \le (2^\omega)^\omega = 2^\omega.$$

For a space X, $Z(X)$ denotes the family of all zero-sets of X. The boundary Bd Z of Z is the set $Z \backslash \mathrm{int}_X Z$.

The following (known) fact will be used; its proof is left to the reader.

4.11. **Lemma.** *Let X be a realcompact space. Then*

(a) *if $p \in \beta X \backslash X$ then $\{p\}$ is not a zero-set of $\beta X \backslash X$;*

(b) *$\beta X \backslash X$ has no isolated elements; and*

(c) *the local weight of $\beta X \backslash X$ at each of its elements is uncountable.*

4.12. **Theorem** (Lemma 3.1 of Fine and Gillman [6]). *If X is a locally compact realcompact space and Z is a zero-set of $\beta X \backslash X$ then $Z = \mathrm{cl}_{\beta X \backslash X}\, \mathrm{int}_{\beta X \backslash X}\, Z$.*

Using Theorem 4.12 and imitating the usual proof of Baire's category theorem the following result is obtained.

4.13. **Corollary.** *If X is a locally compact and realcompact space then $\beta X \backslash X$ has the ω^+-Baire category property.*

The following lemma was noted by the second author in 1963 (unpublished); a more general version is given by Woods [33] (Theorem 2.7).

4.14. **Lemma.** *If X is a locally compact realcompact space and A is a (closed) nowhere dense subset of X then $\mathrm{cl}_{\beta X} A \backslash X$ is nowhere dense in $\beta X \backslash X$.*

Proof. Let $B = \mathrm{cl}_{\beta X} A \backslash X$; thus B is a closed subset of $\beta X \backslash X$. If the interior of B is nonempty there is a nonempty zero-set Z_0 of $\beta X \backslash X$ such that $Z_0 \subset B$. Because X is locally compact it follows that $\beta X \backslash X$ is compact and hence C-embedded in βX; thus there is a zero-set Z_1 of βX such that $Z_0 = Z_1 \cap (\beta X \backslash X)$. If $p \in Z_0$ then, since X is realcompact, there is a zero-set Z_2 of βX such that $p \in Z_2 \subset \beta X \backslash X$. We define $Z = Z_0 \cap Z_2$; thus Z is a nonempty zero-set of βX such that $Z \subset B$. Let $f \in C(\beta X)$ be such that $Z = \{x \in \beta X : f(x) = 0\}$

and $f(x) \geq 0$ for $x \in X$. It is easy to see, using 1.20 of [8] that there is a set $N = \{p_n : n < \omega\}$ such that

N is homeomorphic to ω,

N is a subset of A C-embedded in X, and

$f(p_n) > 0$ for $n < \omega$,

$\lim_{n \to \omega} f(p_n) = 0$.

We choose a compact neighborhood K_n of p_n (in X) such that

$|f(p) - f(p_n)| < 1/n$ if $p \in K_n$ and $n < \omega$

$\{K_n : n < \omega\}$ is locally finite,

$K_n \cap K_m = \emptyset$ if $n < m < \omega$.

Then $\operatorname{int}_X K_n \backslash A \neq \emptyset$ for $n < \omega$, since A is a nowhere dense subset of X. Let $q_n \in \operatorname{int}_X K_n \backslash A$ for $n < \omega$ and $Q = \{q_n : n < \omega\}$. It is easy to see that Q is C-embedded in X (cf. 3L.1 in [8]). Thus, using 3B.2 of [8], we have that $\operatorname{cl}_{\beta X} Q \cap \operatorname{cl}_{\beta X} A = \emptyset$. Finally we have $f(q) = 0$ for $q \in \operatorname{cl}_{\beta X} Q \backslash X$, in contradiction to the fact that $Z \subset B \subset \operatorname{cl}_{\beta X} A$.

4.15. **Theorem.** *Assume that* $\omega^+ = 2^\omega$ *and that* X *is a noncompact locally compact realcompact space such that* $|C(X)| \leq 2^\omega$. *Then each of the spaces* $(\beta X \backslash X)_{\omega^+}$ *and* $P_{\omega^+}(\beta X \backslash X)$ *is homeomorphic to* $(2^{(\omega^+)})_{\omega^+}$.

Proof. We note since $\beta X \backslash X$ is closed, and thus C-embedded, in βX that

$$(*) \qquad w(\beta X \backslash X) \leq |Z(\beta X \backslash X)| \leq |Z(\beta X)| \leq |C(\beta X)| \leq |C(X)| \leq 2^\omega = \omega^+.$$

To prove that $(\beta X \backslash X)_{\omega^+}$ is homeomorphic to the space $(2^{(\omega^+)})_{\omega^+}$ we make use of Theorem 4.1. Thus ω^+ is regular and $\beta X \backslash X$ is compact and it follows from $(*)$ that $w(\beta X \backslash X) \leq \omega^+$; finally if $p \in \beta X \backslash X$ then the local weight of $\beta X \backslash X$ at p is at least ω^+ according to Lemma 4.11 (c). The required homeomorphism follows.

For the second homeomorphism we note that

$$P_{\omega^+}(\beta X \backslash X) = \bigcap \{\beta X \backslash X \backslash \operatorname{Bd} Z : Z \in Z(\beta X \backslash X)\}$$

and that the restriction of the identity function $(\beta X \backslash X)_{\omega^+} \to \beta X \backslash X$ to $P_{\omega^+}(\beta X \backslash X)$ is a homeomorphism. We make use of Theorem 3.5 (b) and the already established homeomorphism between $(\beta X \backslash X)_{\omega^+}$ and $(2^{(\omega^+)})_{\omega^+}$. Since ω^+ is not weakly compact, and, from $(*)$, we have $|Z(\beta X \backslash X)| \leq \omega^+$, we only have to verify that

$$(\beta X \backslash X \backslash \operatorname{Bd} Z)_{\omega^+} \text{ is dense in } (\beta X \backslash X)_{\omega^+} \quad \text{for } Z \in Z(\beta X \backslash X).$$

The family $Z(\beta X \backslash X)$ is a base for the topology of $(\beta X \backslash X)_{\omega^+}$, and thus it is sufficient to verify that

$$(\beta X\setminus X\setminus \mathrm{Bd}\, Z)\cap W\neq\emptyset \quad \text{for } W\in Z(\beta X\setminus X) \text{ and } W\neq\emptyset.$$

Indeed if this relation fails for some $W\in Z(\beta X\setminus X)$ then $W\subset \mathrm{Bd}\, Z$ contradicting the fact that according to Theorem 4.12, $\mathrm{Bd}\, Z$ is nowhere dense while W has nonempty interior. The proof of the theorem is complete.

Most of Theorem 4.15 has been proved in [4]; specifically, Corollary 3.3 of [4] states that $(\beta X\setminus X)_{\omega^+}$ is homeomorphic to $(2^{(\omega^+)})_{\omega^+}$; and Theorem 3.5 in [4] states that $P_{\omega^+}(\beta X\setminus X)$ is homeomorphic to $(2^{(\omega^+)})_{\omega^+}$ under certain assumptions on X which are stronger than those assumed in the statement of the present Theorem 4.15.

Definition. If X is a space and $p\in\beta X\setminus X$ then p is a *remote point* of $\beta X\setminus X$ if p is not in the closure (in βX) of any discrete subset of X. The set of remote points of $\beta X\setminus X$ is denoted by $R(\beta X\setminus X)$.

The existence of remote points of $\beta\mathbf{R}\setminus\mathbf{R}$, where $\mathbf{R}$ is the space of real numbers, was proved, assuming the continuous hypothesis, by Fine and Gillman in [7]. A Baire-category type of proof of P_{ω^+}-points and of remote points of $\beta X\setminus X$ was first provided by the second author of the paper in 1963 [unpublished]. For more information on remote points the reader is referred to [32].

4.16. **Theorem.** *Assume that* $\omega^+=2^{\omega}$ *and that* X *is a noncompact locally compact separable metric space without isolated elements. Then each of the spaces*

$$(R(\beta X\setminus X))_{\omega^+} \text{ and } R(\beta X\setminus X)\cap P_{\omega^+}(\beta X\setminus X)$$

is homeomorphic to $(2^{(\omega^+)})_{\omega^+}$.

Proof. It is clear that X is σ-compact and such that $|C(X)|\leq 2^{\omega}$. Thus from Theorems 4.15 and 4.9 it follows that $(\beta X\setminus X)_{\omega^+}$ is homeomorphic to $(2^{(\omega^+)})_{\omega^+}$.

For the first homeomorphism we note that

$$R(\beta X\setminus X)=\bigcap\{\beta X\setminus X\setminus \mathrm{cl}_{\beta X} D\colon D \text{ is a discrete subset of } X\}.$$

We make use of Theorem 3.5 (b) and the homeomorphism between $(\beta X\setminus X)_{\omega^+}$ and $(2^{(\omega^+)})_{\omega^+}$. If D is a discrete subset of X then D is countable (since every subspace of a separable metric space is separable), and therefore the cardinal of the family of all discrete subsets of X is at most $2^{\omega}=\omega^+$. Furthermore $\mathrm{cl}_X D$ is nowhere dense in X; indeed if there is a nonempty open set V of X such that $V\subset \mathrm{cl}_X D$, then $V\cap D$ is dense in V and since $V\cap D$ is discrete it is locally compact and therefore $V\cap D$ is open in V and hence in X, in contradiction to the assumption that X does not have isolated elements. It follows from Lemma 4.14 (since X is realcompact as well) that $\mathrm{cl}_{\beta X} D\setminus X$ is nowhere dense in $\beta X\setminus X$. We verify that

$$(\beta X \backslash X \backslash \mathrm{cl}_{\beta X} D)_{\omega^+} \quad \text{is dense in } (\beta X \backslash X)_{\omega^+},$$

for every discrete set D of X. The family $Z(\beta X \backslash X)$ is a base for the topology of $(\beta X \backslash X)_{\omega^+}$ and thus it is sufficient to verify that

$$(\beta X \backslash X \backslash \mathrm{cl}_{\beta X} D) \cap Z \neq \emptyset \quad \text{for } Z \in Z(\beta X \backslash X) \text{ and } Z \neq \emptyset.$$

Indeed if this relation fails for some $Z \in Z(\beta X \backslash X)$ then $Z \subset \mathrm{cl}_{\beta X} D$ contradicting the fact that according to Theorem 4.12, Z has nonempty interior.

The second homeomorphism follows from a combination of the proof of the second part of Theorem 4.15 and the proof just given; the details are omitted.

Finally we mention that most of Theorems 4.15 and 4.16 can be proved assuming Martin's axiom instead of the continuum hypothesis. We will not give here the details of the proofs, which involve techniques similar to those used in the proofs of Theorems 4.15 and 4.16 together with some arguments similar to those given in Booth [3]. For information on Martin's axiom and its relation to ultrafilters on ω the reader is referred to [3], [20], [31].

4.17. **Corollary.** *Assume Martin's axiom.*

(a) *If X is a locally compact, σ-compact, noncompact metric space and $A \subset \beta X \backslash X$ with $|A| \leq 2^{\omega}$ then each of the spaces $(\beta X \backslash X)_{2^\omega}$, $P_{2^\omega}(\beta X \backslash X) \backslash A$, $(\mathrm{RK}(\omega))_{2^\omega}$ is homeomorphic to $(2^{(2^\omega)})_{2^\omega}$.*

(b) *If in addition to the properties of X given in* (a), *X does not have isolated elements then each of the spaces $(R(\beta X \backslash X))_{2^\omega}$, $R(\beta X \backslash X) \cap P_{2^\omega}(\beta X \backslash X)$ is homeomorphic to $(2^{(2^\omega)})_{2^\omega}$.*

The following results should be contrasted: if $\omega^+ = 2^{\omega}$ then $\beta\omega \backslash \omega$ is homeomorphic to the Stone space of the 2^{ω}-homogeneous-universal Boolean algebra $S(2^{\omega})$ of cardinality 2^{ω} (cf. Parovičenko [26] and Keisler [14]). However, if the continuum hypothesis fails then $\beta\omega \backslash \omega$ is definitely not homeomorphic to $S(2^{\omega})$, according to a result of Hausdorff [9](2), even though if Martin's axiom holds as well, then Corollary 4.17 (a) above implies that $(\beta\omega \backslash \omega)_{2^\omega}$ *is* homeomorphic to $(S(2^{\omega}))_{2^\omega}$ (since $2^{\omega 2^{\omega}} = 2^{\omega}$ and both spaces are homeomorphic to $(2^{(2^\omega)})_{2^\omega}$).

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DEPARTMENT OF MATHEMATICS, MCGILL UNIVERSITY, MONTREAL, QUEBEC, CANADA (Current address of S. Negrepontis)

DEPARTMENT OF MATHEMATICS, SIR GEORGE WILLIAMS UNIVERSITY, MONTREAL, QUEBEC, CANADA (Current address of H. H. Hung)

DEPARTMENT OF MATHEMATICS, ATHENS UNIVERSITY, ATHENS, GREECE